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Holomorphic vector bundles and non-perturbative vacua in M-theory

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Abstract: We review the spectral cover formalism for constructing both U(n) and SU(n) holomorphic vector bundles on elliptically fibered Calabi-Yau three-folds which admit a section. We discuss the allowed bases of these three-folds and show that physical constraints eliminate Enriques surfaces from consideration. Relevant properties of the remaining del Pezzo and Hirzebruch surfaces are presented. Restricting the structure group to SU(n), we derive, in detail, a set of rules for the construction of three-family particle physics theories with phenomenologically relevant gauge groups. We show that anomaly cancellation generically requires the existence of non-perturbative vacua containing five-branes. We illustrate these ideas by constructing four explicit three-family non-perturbative vacua.

Keywords: M-Theory, Superstring Vacua, Differential and Algebraic Geometry, Compactification and String Models.

JHEP06(1999)034

Contents

1. Introduction

The ground breaking work of Hořava and Witten [1, 2] showed that $N = 1$ supersymmetric, chiral theories can arise in four-dimensions upon compactification of M-theory on an $S^1/\mathbb{Z}_2$ orbifold times a Calabi-Yau three-fold. Early work on this subject indicated that one could get reasonable phenomenological values for Newton's constant and the gauge unification parameter and scale [3, 4]. Interestingly, acceptable values were contingent upon the radius of the orbifold being an order of magnitude, or more, larger than the Calabi-Yau radius. Thus, with decreasing energy, the universe appears first eleven-, then five- and, finally, four-dimensional. The effective four-dimensional reduction of Hořava-Witten theory was first constructed, directly from eleven-dimensions, in [5, 6]. Various aspects of this four-dimensional theory have been discussed by many authors [7]–[27].

More recently, the effective five-dimensional heterotic M-theory was constructed [28, 29]. It was shown to be a specific form of gauged $N = 1$ supergravity coupled to hyper and vector supermultiplets and bounded by two four-dimensional orbifold fixed planes. These boundary planes contain $N = 1$ supersymmetric gauge theories coupled to chiral matter supermultiplets. In [28, 29], the so-called standard embedding of the spin connection into an SU(3) subgroup of E_8 on one of the orbifold planes was assumed. It was shown in [28, 29], that this five-dimensional theory does not admit flat space as its static vacuum solution. Rather, it supports BPS three-brane domain walls, much as the gauged type IIA supergravity theory discussed by Romans [30] supports BPS eight-branes [31]. The minimal number of such domain walls is two, with one wall located at each orbifold fixed plane. When expanded to leading non-trivial order, this pair of BPS three-branes exactly reproduces the eleven-dimensional "deformations" of the metric line element discussed by Witten [3]. We refer to this as the minimal, or perturbative, vacuum. This five-dimensional theory is not simply a formal development, since the universe actually passes through this five-dimensional phase for an energy range of an order of magnitude or so below the unification scale. Various physical aspects of this theory have been discussed in [32, 33]. It was shown in [28, 29] that, when dimensionally reduced onto the worldvolume of the pair of three-branes, the five-dimensional theory exactly reproduces the effective four-dimensional theory derived by other methods in [5, 6], as it must.

As emphasized in [34], the restriction to the standard embedding of the spin connection into the gauge connection when reducing heterotic M-theory to five- and four-dimensions is very unnatural. This is so because, unlike the case of the weakly coupled heterotic superstring, there is no choice of embedding, standard or otherwise, that is to say no choice of gauge field background, that allows one to set the entire supergravity three-form to zero. Hence, in heterotic M-theory, one should consider arbitrary gauge field backgrounds which preserve $N = 1$ supersymmetry. This was also discussed in [35]. In addition, as was first noted in [3], one can include M5-branes in the background and still preserve supersymmetry, provided the branes are wrapped on holomorphic curves within the Calabi-Yau threefold. These general vacua, which in five dimensions involving extra BPS three-branes, the remnants of the five-branes, in addition to the two located at the orbifold planes, were also analyzed in [34]. It was shown that the worldvolume theories of the extra three-branes are $N = 1$ supersymmetric gauge theories, whose gauge groups depend on the genus and the position in their moduli space of the holomorphic curves. We refer to five-dimensional vacua with extra BPS three-branes as non-perturbative vacua. The main conclusion of [34] was that, because of the condition of anomaly cancellation, including background five-branes greatly relaxes the constraint on allowed non-standard embeddings, and allows much more freedom in constructing vacua. It should be noted that there is a long history to constructing backgrounds with non-standard embeddings ($(0, 2)$ models), such as, for example, [40] and the three-family model of [41], or more recently of [42]. Furthermore, the first explicit construction of the gauge couplings for a non-standard embedding in an M-theory context was given in [43]. The background was a Calabi-Yau orbifold without fivebranes and the results were then compared with the couplings in the weakly coupled heterotic theory. The paper also contained an interesting discussion of the effect of five-branes on the gauge couplings.

The results of [34] indicated the importance of heterotic M-theories with non-standard embeddings and non-perturbative vacua, analyzing the general structure of such backgrounds. A specific example of these vacua was given recently in [36], where explicit constructions were carried out within the context of holomorphic vector bundles on the orbifold planes of heterotic M-theory compactified on elliptically fibered Calabi-Yau three-folds which admit a section. The results of [36] rely upon recent mathematical work by Friedman, Morgan and Witten [37], Donagi [38] and Bershadsky, Johansen, Pantev, and Sadov [39] who show how to explicitly construct such vector bundles, and on results of [44, 45] who computed the family generation index in this context. Most recently work has also appeared discussing the stability of this index under deformations of the bundle [46]. Extending these results, we were able to formulate rules for constructing three-family particle physics theories with phenomenologically interesting gauge groups. As expected, the appearance of gauge groups other than the E_6 group of the standard embedding, as well as the three-family condition, necessitate the existence of M5-branes and, hence, non-perturbative vacua. In [36], we showed how to compute the topological class of these five-branes and, given this class, analyzed an example of the moduli space of the associated holomorphic curves. Our results were summarized as a set of rules for constructing vacua. In addition, we gave one concrete example of a three-family model with gauge group $\mathrm{SU}(5)$, along with its five-brane class and moduli space.

In this paper, we greatly enlarge the discussion of the results in [36], deriving in detail the rules presented there. In order to make this work more accessible to physicists, as well as to lay the foundation for the necessary derivations and proofs, we present brief discussions of (1) elliptically fibered Calabi-Yau three-folds, (2) spectral cover constructions of both $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ bundles, (3) Chern classes, and (4) complex surfaces, specifically del Pezzo, Hirzebruch, and Enriques surfaces. Using this background, we explicitly derive the rules for the construction of three-family models based on semi-stable holomorphic vector bundles with structure group $\mathrm{SU}(n)$. Specifically, we construct the form of the five-brane class $[W]$, as well as the con-

JHEP06(1999)034

straints imposed on this class due to the three-family condition, the restriction that the vector bundle have structure group $\mathrm{SU}(n)$ and the requirement that $[W]$ be an effective class. From these considerations, we derive a set of rules that are presented in section 7. As discussed in this paper, elliptically fibered Calabi-Yau three-folds that admit a section can only have del Pezzo, Hirzebruch, Enriques, and blown-up Hirzebruch surfaces as a base. We show in section 8, however, that Enriques surfaces can never lead to effective five-brane curves in vacua with three generations. Therefore, the base B of the elliptic fibration is restricted to be a del Pezzo, Hirzebruch or a blow-up of a Hirzebruch surface. In appendix B, we present the generators of all effective classes in $H_2(B, \mathbb{Z})$, as well as the first and second Chern classes $c_1(B)$ and $c_2(B)$, for these allowed bases. Combining the rules in section 7 with the generators and Chern classes given in appendix B, we present a general algorithm for the construction of non-perturbative vacua corresponding to three-family particle physics theories with phenomenologically relevant gauge groups. We illustrate this algorithm by constructing four such non-perturbative vacua, three with del Pezzo surfaces as a base and one with a Hirzebruch surface. We do not, in this paper, discuss the moduli spaces of the five-brane holomorphic curves. An explicit example of such a moduli space was given in [36]. A discussion of the general method for constructing five-brane curve moduli spaces will be presented, in detail, elsewhere [47]. We also leave for later discussions of some natural phenomenological questions, in particular the breaking of the gauge group down to the standard model, since the main focus of this paper is to present the tools for constructing non-perturbative vacua. The results given here are all within the context of heterotic M-theory. However, our formalism will apply, with very minor modifications, to compactifications of the weakly coupled heterotic superstring on elliptically fibered Calabi-Yau three-folds with NS5-branes. Finally, we would like to point out that the five-brane classes $[W]$ that appear naturally in three-family models have a component in the base surface. That is, they are not wrapped purely on the fiber and, therefore, are not dual to three-branes in F-theory. The interesting question of what vacua they are dual to in F-theory and superstrings will be discussed elsewhere. In the context of toric varieties this has been addressed in an interesting paper by Rajesh [48].

2. Holomorphic gauge bundles, five-branes, non-perturbative vacua

In this section, we will discuss the generic properties of heterotic M-theory vacua appropriate for a reduction of the theory to $N = 1$ supersymmetric theories in both five and four dimensions. The M-theory vacuum is given in eleven dimensions by specifying the metric g_{IJ} and the three-form C_{IJK} with field strength $G_{IJKL} = 24\partial_{[I}C_{JKL]}$ of the supergravity multiplet. Following Hořava and Witten [1, 2] and

Witten [3], the space-time structure, to lowest order in the expansion parameter $\kappa^{2/3}$, will be taken to be

$$M_{11} = M_4 \times \frac{S^1}{\mathbb{Z}_2} \times X\,, \tag{2.1}$$

where M_4 is four-dimensional Minkowski space, $S^1/\mathbb{Z}_2$ is a one-dimensional orbifold and X is a smooth Calabi-Yau three-fold. The vacuum space-time structure becomes more complicated at the next order in $\kappa^{2/3}$, but this metric "deformation", which has been the subject of a number of papers [3, 4, 5], can be viewed as arising as the static vacuum of the five-dimensional effective theory [28, 29] and, hence, need not concern us here.

The $\mathbb{Z}_2$ orbifold projection necessitates the introduction, on each of the two ten-dimensional orbifold fixed planes, of an $N=1$, E_8 Yang-Mills supermultiplet which is required for anomaly cancellation. In general, one can consider vacua with non-zero gauge fields excited within the Calabi-Yau space, on each plane. However, the supersymmetry transformations imply the fields must be a solution of the hermitian Yang-Mills equations for an E_8-valued connection in order to be compatible with four preserved supercharges in four dimensions. Donaldson [49] and Uhlenbeck and Yau [50] have shown that picking a solution of the hermitian Yang-Mills equations is equivalent to the topological problem of choosing a semi-stable, holomorphic bundle with the structure group being the complexification $E_{8\mathbf{C}}$ of E_8. It is this second formulation we will use in this paper and we will often refer to fixing the background gauge fields as simply choosing a gauge bundle. In the following, we will denote both the real and complexified groups by E_8, letting context dictate which group is being referred to. (In general, we will denote any group G and its complexification $G_{\mathbf{C}}$ simply as G). These semi-stable, holomorphic gauge bundles are, a priori, allowed to be arbitrary in all other respects. In particular, there is no requirement that the spin-connection of the Calabi-Yau three-fold be embedded into an SU(3) subgroup of the gauge connection of one of the E_8 bundles, the so-called standard embedding. This generalization to arbitrary semi-stable holomorphic gauge bundles is what is referred to as non-standard embedding. The terms standard and non-standard embedding are historical and somewhat irrelevant in the context of M-theory, where no choice of embedding can ever set the entire three-form C_{IJK} to zero. For this reason, we will avoid those terms and simply refer to arbitrary semi-stable holomorphic E_8 gauge bundles. Fixing the gauge bundle will in general completely break the E_8 gauge symmetry in the low-energy theory. However, it is clear, that in order to preserve a non-trivial low-energy gauge group, we can restrict the transition functions to be elements of any subgroup G of E_8, such as $G = \mathrm{U}(n)$, $\mathrm{SU}(n)$, or $\mathrm{Sp}(n)$. We will refer to the restricted bundle as a semi-stable, holomorphic G bundle, or simply as a G bundle. It is clear that the G_1 bundle on one orbifold plane and the G_2 bundle on the other plane need not, generically, have the same subgroups G_1 and G_2 of E_8. We will denote the semi-stable holomorphic gauge bundle on the ith orbifold plane by V_i and the associated structure group by G_i.

In addition, as discussed in [3, 34, 36], we will allow for the presence of five-branes located at points throughout the orbifold interval. The five-branes will preserve $N = 1$ supersymmetry provided they are wrapped on holomorphic two-cycles within X and otherwise span the flat Minkowski space M_4 [3, 51, 52]. The inclusion of five-branes is essential for a complete discussion of M-theory vacua. The reason for this is that, given a Calabi-Yau three-fold background, the presence of five-branes allows one to construct large numbers of gauge bundles that would otherwise be disallowed [34, 36].

The requirements of gauge and gravitational anomaly cancellation on the two orbifold fixed planes, as well as anomaly cancellation on each five-brane worldvolume, places a further very strong constraint on, and relationship between, the space-time manifold, the gauge bundles and the five-brane structure of the vacuum. Specifically, anomaly cancellation necessitates the addition of magnetic sources to the four-form field strength Bianchi identity. The modified Bianchi identity is given by

$$(dG)_{11\bar{I}\bar{J}\bar{K}\bar{L}} = 2\sqrt{2}\pi\left(\frac{\kappa}{4\pi}\right)^{2/3}\Big[2J^{(0)}\delta\big(x^{11}\big) + 2J^{(N+1)}\delta\big(x^{11}-\pi\rho\big) + \\ + \sum_{n=1}^{N} J^{(n)}\big(\delta\big(x^{11}-x_n\big)+\delta\big(x^{11}+x_n\big)\big)\Big]_{\bar{I}\bar{J}\bar{K}\bar{L}}. \tag{2.2}$$

The sources $J^{(0)}$ and $J^{(N+1)}$ on the orbifold planes are

$$J^{(0)} = -\frac{1}{16\pi^2}\left(\operatorname{tr} F^{(1)}\wedge F^{(1)} - \frac{1}{2}\operatorname{tr} R\wedge R\right)\Big|_{x^{11}=0} \tag{2.3}$$

and

$$J^{(N+1)} = -\frac{1}{16\pi^2}\left(\operatorname{tr} F^{(2)}\wedge F^{(2)} - \frac{1}{2}\operatorname{tr} R\wedge R\right)\Big|_{x^{11}=0} \tag{2.4}$$

respectively. The two-form $F^{(i)}$ is the field strength of a connection on the gauge bundle V_i of the ith orbifold plane and R is the curvature two-form on the Calabi-Yau three-fold. By "tr" for the gauge fields we mean (1/30)th of the trace in the **248** representation of E_8, while for the curvature it is the trace in the fundamental representation of the tangent space SO(10). We have also introduced N additional sources $J^{(n)}$, where $n = 1,\ldots,N$. These arise from N five-branes located at $x^{11} = x_1,\ldots,x_N$ where $0 \le x_1 \le \cdots \le x_N \le \pi\rho$. Note that each five-brane at $x = x_n$ has to be paired with a mirror five-brane at $x = -x_n$ with the same source since the Bianchi identity must be even under the $\mathbb{Z}_2$ orbifold symmetry. These sources are four-form delta functions localized on the fivebrane worldvolume. As forms there are Poincaré dual to the six-dimensional cycles of the fivebrane worldvolumes. (This duality is summarized in appendix A.) In particular their normalization is such that there are in integer cohomology classes.

Non-zero source terms on the right hand side of the Bianchi identity (2.2) preclude the simultaneous vanishing of all components of the three-form C_{IJK}. The result of this is that, to next order in the Hořava-Witten expansion parameter $\kappa^{2/3}$, the space-time of the supersymmetry preserving vacua gets "deformed" away from that given in expression (2.1). As discussed above, this deformation of the vacuum need not concern us here. In this paper, we will focus on yet another aspect of the Bianchi identity (2.2), a topological condition that constrains the cohomology of the vacuum. This constraint is found as follows. Consider integrating the Bianchi identity (2.2) over any five-cycle which spans the orbifold interval together with an arbitrary four-cycle $\mathcal{C}_4$ in the Calabi-Yau three-fold. Since dG is exact, this integral must vanish. Physically, this is the statement that there can be no net charge in a compact space, since there is nowhere for the flux to "escape". Performing the integral over the orbifold interval, we derive, using (2.2), that

$$\sum_{n=0}^{N+1} \int_{\mathcal{C}_4} J^{(n)} = 0\,. \tag{2.5}$$

Hence, the total magnetic charge over $\mathcal{C}_4$ vanishes. Since this is true for an arbitrary four-cycle $\mathcal{C}_4$ in the Calabi-Yau three-fold, it follows that the sum of the sources must be cohomologically trivial. That is

$$\left[\sum_{n=0}^{N+1} J^{(n)}\right] = 0\,. \tag{2.6}$$

(Throughout this paper, we will use the notation $[\omega]$ to refer to the cohomology class of ω, in this case a closed four-form.) The physical meaning of this expression becomes more transparent if we rewrite it using eqs. (2.3) and (2.4). Using these expressions, eq. (2.6) becomes

$$-\frac{1}{16\pi^2}\left[\operatorname{tr} F^{(1)} \wedge F^{(1)}\right] - \frac{1}{16\pi^2}\left[\operatorname{tr} F^{(2)} \wedge F^{(2)}\right] + \frac{1}{16\pi^2}\left[\operatorname{tr} R \wedge R\right] + \sum_{n=1}^{N}\left[J^{(n)}\right] = 0\,. \tag{2.7}$$

It is useful to recall that the second Chern class of an arbitrary G bundle V, thought of as an E_8 sub-bundle, is defined to be

$$c_2(V) = -\frac{1}{16\pi^2}\left[\operatorname{tr} F \wedge F\right]. \tag{2.8}$$

Similarly, the second Chern class of the tangent bundle of the Calabi-Yau manifold X is given by

$$c_2(TX) = -\frac{1}{16\pi^2}\left[\operatorname{tr} R \wedge R\right]\,, \tag{2.9}$$

JHEP06(1999)034

where as above the trace is taken in the vector representation of SO(6) $\supset$ SU(3). It follows that expression (2.7) can be written as

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)\,, \tag{2.10}$$

where

$$[W] = \sum_{n=1}^{N} \left[J^{(n)}\right] \tag{2.11}$$

is the four-form cohomology class associated with the five-branes. This is a fundamental constraint imposed on the vacuum structure. We will explore this cohomology condition in great detail in this paper. Since the Chern classes are integer, this is a condition between integer classes (or rather the image of $H^4(X,\mathbb{Z})$ in $H^4_{\mathrm{DR}}(X,\mathbb{R})$ as derived). This means that integrating this constraint over an arbitrary four-cycle $\mathcal{C}_4$ yields the integral expression

$$n_1(\mathcal{C}_4) + n_2(\mathcal{C}_4) + n_5(\mathcal{C}_4) = n_R(\mathcal{C}_4) \tag{2.12}$$

which states that the sum of the number of gauge instantons on the two orbifold planes, plus the sum of the five-brane magnetic charges, must equal the instanton number for the Calabi-Yau tangent bundle, a number which is fixed once the Calabi-Yau three-fold is chosen. Note that the normalization is such that one unit of five-brane charge is equal to one unit of instanton charge.

To summarize, we are considering vacuum states of M-theory with the following structure.

- Space-time is taken to have the form

$$M_{11} = M_4 \times \frac{S^1}{\mathbb{Z}_2} \times X\,, \tag{2.13}$$

  where X is a Calabi-Yau three-fold.

- There is a semi-stable holomorphic gauge bundle V_i with fiber group $G_i \subseteq E_8$ over the Calabi-Yau three-fold on the ith orbifold fixed plane for $i = 1, 2$. The structure groups G_1 and G_2 of the two bundles can be any subgroups of E_8 and need not be the same.

- We allow for the presence of five-branes in the vacuum, which are wrapped on holomorphic two-cycles within X and are parallel to the orbifold fixed planes.

- The Calabi-Yau three-fold, the gauge bundles and the five-branes are subject to the cohomological constraint on X

$$c_2(V_1) + c_2(V_2) + [W] = c_2(TX)\,, \tag{2.14}$$

where $c_2(V_i)$ and $c_2(TX)$ are the second Chern classes of the gauge bundle V_i and the tangent bundle TX respectively and $[W]$ is the class associated with the five-branes.

Vacua of this type will be referred to as non-perturbative heterotic M-theory vacua.

The discussion given in this section is completely generic, in that it applies to any Calabi-Yau three-fold and any gauge bundles that can be constructed over it. However, realistic particle physics theories require the explicit construction of these gauge bundles. Until now, such constructions have been carried out for the restricted cases of standard or non-standard embeddings without five-branes. These restrictions make it very difficult to obtain realistic particle physics theories, that is, theories with three families, appropriate gauge groups and so on. It is the purpose of this paper to resolve these difficulties by explicitly constructing theories with non-perturbative heterotic M-theory vacua, utilizing the freedom introduced by including five-branes.

Specifically, we will present a formalism for the construction of semi-stable holomorphic gauge bundles with fiber groups G_1 and G_2 over the two orbifold fixed planes. In this paper, for specificity, we will restrict the structure groups to be

$$G_i = \mathrm{U}(n_i) \qquad \text{or} \qquad \mathrm{SU}(n_i)\,, \tag{2.15}$$

for $i = 1, 2$. Other structure groups, such as $\mathrm{Sp}(n)$ or exceptional groups, will be discussed elsewhere. Our explicit bundle constructions will be achieved over the restricted, but rich, set of elliptically fibered Calabi-Yau three-folds which admit a section. Such three-folds have been extensively discussed within the context of duality between string theory and M- and F-theory. Independent of this use, however, elliptically fibered Calabi-Yau three-folds with a section are known to be the simplest class of Calabi-Yau spaces on which one can explicitly construct bundles, compute Chern classes, moduli spaces and so on [37, 38, 39]. This makes them a compelling choice for the construction of concrete particle physics theories. Having constructed the bundles, one can explicitly calculate the gauge bundle Chern classes $c_2(V_i)$ for $i = 1, 2$, as well as the tangent bundle Chern class $c_2(TX)$. Having done so, one can then find the class $[W]$ of the five-branes using the cohomology condition (2.10). That is, in this paper we will present a formalism in which the structure of non-perturbative M-theory vacua can be calculated.

As will be discussed in detail below, having constructed a non-perturbative vacuum, we can compute the number of low energy families and the Yang-Mills gauge group associated with that vacuum. We will show that, because of the flexibility introduced by the presence of five-branes, we will easily construct non-perturbative vacua with three-families. Similarly, one easily finds phenomenologically interesting gauge groups, such as E_6, $\mathrm{SU}(5)$, and $\mathrm{SO}(10)$, as the E_8 subgroups commutant with the G-bundle structure groups, such as $\mathrm{SU}(3)$, $\mathrm{SU}(4)$, and $\mathrm{SU}(5)$ respectively, on the observable orbifold fixed plane. In addition, using the cohomology constraint (2.10), one can explicitly determine the cohomology class $[W]$ of the five-branes for a specific vacuum. Hence, one can compute the holomorphic curve associated with the five-branes exactly and determine all of its geometrical attributes. These include the number of its irreducible components, which tells us the number of *independent* five-branes, and its genus, which will tell us the minimal gauge group on the five-brane worldvolume when dimensionally reduced on the holomorphic curve. Furthermore, we are, in general, able to compute the entire moduli space of the holomorphic curve. This can tell us about gauge group enhancement on the five-brane worldvolume, for example. In [36], we discussed the generic properties of examples of holomorphic curves associated with five-branes. We will present a more detailed discussion in [47].

Finally, we want to point out that there are more moduli associated with these non-perturbative vacua. These are (1) the moduli associated with the gauge instantons on the two orbifold planes and (2) the translation moduli of the five-branes in the orbifold dimension. Taken along with the five-brane holomorphic curve moduli, these form an enormously complicated, but physically rich, space of non-perturbative vacua. The structure of the full moduli space of non-perturbative heterotic M-theory vacua will be discussed elsewhere.

3. Elliptically fibered Calabi-Yau three-folds

As discussed previously, we will consider non-perturbative vacua where the Calabi-Yau three-fold is an elliptic fibration which admits a section. In this section, we give an introduction to these spaces, summarizing the properties we will need in order to compute explicitly properties of the vacua.

An elliptically fibered Calabi-Yau three-fold X consists of a base B, which is a complex two-surface, and an analytic map

$$\pi : X \longrightarrow B \tag{3.1}$$

with the property that for a generic point $b \in B$, the fiber

$$E_b = \pi^{-1}(b) \tag{3.2}$$

is an elliptic curve. That is, E_b is a Riemann surface of genus one. In addition, we will require that there exist a global section, denoted σ, defined to be an analytic map

$$\sigma : B \longrightarrow X \tag{3.3}$$

that assigns to every point $b \in B$ the zero element $\sigma(b) = p \in E_b$ discussed below. The requirement that the elliptic fibration have a section is crucial for duality to F-theory and to make contact with the Chern class formulas in [37]. However, this assumption does not seem fundamentally essential and we will explore bundles without sections in future work [53]. The Calabi-Yau three-fold must be a complex Kähler manifold. This implies that the base is itself a complex manifold, while we have already assumed that the fiber is a Riemann surface and so has a complex structure. Furthermore, the fibration must be holomorphic, that is, it must have holomorphic transition functions. Finally, the condition that the Calabi-Yau three-fold has vanishing first Chern class puts a further constraint on the types of fibration allowed.

Let us start by briefly summarizing the properties of an elliptic curve E. It is a genus one Riemann surface and so can be embedded in the two-dimensional complex projective space $\mathbb{P}^2$. A simple way to do this is by using the homogeneous Weierstrass equation

$$zy^2 = 4x^3 - g_2xz^2 - g_3z^3 \,, \tag{3.4}$$

where x, y, and z are complex homogeneous coordinates on $\mathbb{P}^2$. It follows that we identify $(\lambda x, \lambda y, \lambda z)$ with (x, y, z) for any non-zero complex number λ. The parameters g_2 and g_3 encode the different complex structures one can put on the torus. Provided $z \neq 0$, we can rescale to affine coordinates where $z = 1$. We then see, viewed as a map from x to y, that there are two branch cuts in the x-plane, linking $x = \infty$ and the three roots of the cubic equation $4x^3 - g_2x - g_3 = 0$. When any two of these points coincide, the elliptic curve becomes singular. This corresponds to one of the cycles in the torus shrinking to zero. Such singular behaviour is characterized by the discriminant

$$\Delta = g_2^3 - 27g_3^2 \tag{3.5}$$

vanishing. Finally, we note that the complex structure provides a natural notion of addition of points on the elliptic curve. The torus can also be considered as the complex plane modulo a discrete group of translations. Addition of points in the complex plane then induces a natural notion of addition of points on the torus. Translated to the Weierstrass equation, the identity element corresponds to the point where x/z and y/z become infinite. Thus, in affine coordinates, the element $p \in E$ is the point $x = y = \infty$. This can be scaled elsewhere in non-affine coordinates, such as to $x = z = 0$, $y = 1$.

The elliptic fibration is defined by giving the elliptic curve E over each point in the base B. If we assume the fibration has a global section, and in this paper we do, then on each coordinate patch this requires giving the parameters g_2 and g_3 in the Weierstrass equation as functions on the base. Globally, g_2 and g_3 will be sections of appropriate line bundles on B. In fact, specifying the type of an elliptic fibration over B is equivalent to specifying a line bundle on B. Given the elliptic fibration $\pi : X \to B$, we define $\mathcal{L}$ as the line bundle on B whose fiber at $b \in B$ is the cotangent line $T_p(E_b)$ to the elliptic curve at the origin. That is, $\mathcal{L}$ is the conormal bundle to the section $\sigma(B)$ in X. Conversely, given $\mathcal{L}$, we take x and y to scale as sections of $\mathcal{L}^2$ and $\mathcal{L}^3$ respectively, which means that g_2 and g_3 should be sections of $\mathcal{L}^4$ and $\mathcal{L}^6$. By $\mathcal{L}^i$ we mean the tensor product of the line bundle $\mathcal{L}$ with itself i times. In conclusion, we see that the elliptic fibration is characterized by a line bundle $\mathcal{L}$ over the base B together with a choice of sections g_2 and g_3 of $\mathcal{L}^4$ and $\mathcal{L}^6$.

Note that the set of points in the base over which the fibration becomes singular is given by the vanishing of the discriminant $\Delta = g_2^3 - 27g_3^2$. It follows from the above discussion that Δ is a section of the line bundle $\mathcal{L}^{12}$. The zeros of Δ then naturally define a divisor, which in this case is a complex curve, in the base. Since Δ is a section of $\mathcal{L}^{12}$, the cohomology class of the discriminant curve is 12 times the cohomology class of the divisors defined by sections of $\mathcal{L}$.

Finally, we come to the important condition that on a Calabi-Yau three-fold X the first Chern class of the tangent bundle T_X must vanish. The canonical bundle K_X is the line bundle constructed as the determinant of the holomorphic cotangent bundle of X. The condition that

$$c_1(T_X) = 0 \tag{3.6}$$

implies that $K_X = \mathcal{O}$, where $\mathcal{O}$ is the trivial bundle. This, in turn, puts a constraint on $\mathcal{L}$. To see this, note that the adjunction formula tells us that, since B is a divisor of X, the canonical bundle K_B of B is given by

$$K_B = K_X|_B \otimes N_{B/X}\,, \tag{3.7}$$

where $N_{B/X}$ is the normal bundle of B in X and by $K_X|_B$ we mean the restriction of the canonical bundle K_X to the base B. From the above discussion, we know that

$$N_{B/X}^{-1} = \mathcal{L}\,, \qquad K_X|_B = \mathcal{O}\,. \tag{3.8}$$

Inserting this into (3.7) tells us that

$$\mathcal{L} = K_B^{-1}\,. \tag{3.9}$$

This condition means that K_B^{-4} and K_B^{-6} must have sections g_2 and g_3 respectively. Furthermore, the Calabi-Yau property imposes restrictions on how the curves where these sections vanish are allowed to intersect. It is possible to classify the surfaces on which K_B^{-4} and K_B^{-6} have such sections. These are found to be [54] the del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces. In this paper we will discuss the first three possibilities in detail.

As noted previously, in order to discuss the anomaly cancellation condition, we will need the second Chern class of the holomorphic tangent bundle of X. Friedman, Morgan and Witten [37] show that it can be written in terms of the Chern classes of the holomorphic tangent bundle of B as

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)\,, \tag{3.10}$$

where the wedge product is understood, $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of B respectively and σ is the two-form Poincare dual to the global section. We have used the fact that

$$c_1(\mathcal{L}) = c_1\left(K_B^{-1}\right) = c_1(B) \tag{3.11}$$

in writing (3.10).

4. Spectral cover constructions

In this section, we follow the construction of semi-stable holomorphic bundles on elliptically fibered Calabi-Yau manifolds presented in [37, 38, 39]. The idea is to understand the bundle structure on a given elliptic fiber and then to patch these bundles together over the base. The authors in [37, 38, 39] discuss a number of techniques for constructing bundles with different gauge groups. Here we will restrict ourselves to $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ sub-bundles of E_8. These are sufficient to give suitable phenomenological gauge groups. This restriction allows us to consider only the simplest of the different constructions, namely that via spectral covers. In this section, we will summarize the spectral cover construction, concentrating on the properties necessary for an explicit discussion of non-perturbative vacua. We note that for structure groups $G \neq \mathrm{U}(n)$ or $\mathrm{SU}(n)$, the construction of bundles is more complicated than the construction of rank n vector bundles presented here.

As we have already mentioned, the condition of supersymmetry requires that the E_8 gauge bundles admit a field strength satisfying the hermitian Yang-Mills equations. Donaldson, Uhlenbeck, and Yau [49, 50] have shown that this is equivalent to the topological requirement that the associated bundle be semi-stable, with transition functions in the complexification of the gauge group. Since we are considering $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ sub-bundles, this means $\mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SU}(n)_{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ respectively. The spectral cover construction is given in terms of this latter formulation of the supersymmetry condition. Note that the distinction between semi-stable and stable bundles corresponds to whether the hermitian Yang-Mills field strength is reducible or not. This refers to whether, globally, it can be diagonalized into parts coming from different subgroups of the full gauge group. More precisely, it refers to whether or not the holonomy commutes with more that just the center of the group. Usually, a generic solution of the hermitian Yang-Mills equations corresponds to a stable bundle. However, on some spaces, for instance on an elliptic curve, the generic case is semi-stable.

JHEP06(1999)034

4.1 $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ bundles over an elliptic curve

We begin by considering semi-stable bundles on a single elliptic curve E. A theorem of Looijenga [55] states that the moduli space of such bundles for any simply-connected group of rank r is an r-dimensional complex weighted projective space. For the simply-connected group $\mathrm{SU}(n)$, this moduli space is the projective space $\mathbb{P}^{n-1}$. $\mathrm{U}(n)$ is not simply-connected. $\mathrm{U}(n)$ bundles have a discrete integer invariant, their degree or first Chern class, which we denote by d. Let k be the greatest common divisor of d and n. It can be shown that the moduli space of a $\mathrm{U}(n)$ bundle of degree d over a single elliptic curve E is the kth symmetric product of E, denoted by $E^{[k]}$. In this paper, we will restrict our discussion to $\mathrm{U}(n)$ bundles of degree zero. For these bundles, the moduli space is $E^{[n]}$.

A holomorphic $\mathrm{U}(n)_{\mathbb{C}} = \mathrm{GL}(n, \mathbb{C})$ bundle V over an elliptic curve E is a rank n complex vector bundle. As discussed earlier, we will denote $\mathrm{U}(n)_{\mathbb{C}}$ simply as $\mathrm{U}(n)$, letting context dictate which group is being referred to. To define the bundle, we need to specify the holonomy; that is, how the bundle twists as one moves around in the elliptic curve. The holonomy is a map from the fundamental group π_1 of the elliptic curve into the gauge group. Since the fundamental group of the torus is Abelian, the holonomy must map into the maximal torus of the gauge group. This means we can diagonalize all the transition functions, so that V becomes the direct sum of line bundles (one-dimensional complex vector bundles)

$$V = \mathcal{N}_1 \oplus \cdots \oplus \mathcal{N}_n \,. \tag{4.1}$$

Furthermore, the Weyl group permutes the diagonal elements, so that V only determines the $\mathcal{N}_i$ only up to permutations. To reduce from a $\mathrm{U}(n)$ bundle to an $\mathrm{SU}(n)$ bundle, one imposes the additional condition that the determinant of the transition functions be taken to be unity. This implies that the product, formed by simply taking the product of the transition functions for each bundle, satisfies

$$\mathcal{N}_1 \otimes \cdots \otimes \mathcal{N}_n = \mathcal{O} \,, \tag{4.2}$$

where $\mathcal{O}$ is the trivial bundle on E.

The semi-stable condition implies that the line bundles $\mathcal{N}_i$ are of the same degree, which can be taken to be zero. We can understand this from the hermitian Yang-Mills equations. On a Riemann surface, these equations imply that the field strength is actually zero. Thus, the first Chern class of each of the bundles $\mathcal{N}_i$ must vanish, or equivalently $\mathcal{N}_i$ must be of degree zero. On an elliptic curve, this condition means that there is a unique point Q_i on E such that there is a meromorphic section of $\mathcal{N}_i$ which is allowed to have a pole at Q_i and is zero at the origin p. We can write this as

$$\mathcal{N}_i = \mathcal{O}(Q_i) \otimes \mathcal{O}(p)^{-1} \,. \tag{4.3}$$

Let us briefly clarify this notation. One can associate a line bundle to any divisor in a complex manifold X. A divisor is a subspace defined locally by the vanishing of a single holomorphic function or any linear combination of such subspaces $D = \sum_i a_i V_i$ with integer coefficients a_i, and so is a space of one complex dimension lower than X. The associated line bundle has a section corresponding to a meromorphic function on X with poles of order $\leq a_i$ on the subspaces V_i. (If a_i is negative then the function has a zero of order $\geq -a_i$ on V_i.) By $\mathcal{O}(D)$ we mean the line bundle associated with the divisor D. In particular, on a Riemann surface, the divisors are collections of points and for example $\mathcal{O}(Q_i)$ means the line bundle with a section corresponding to a meromorphic function which is allowed to have a first order pole at the particular point Q_i.

If one further restricts the structure group to be $\mathrm{SU}(n)$, then condition (4.2) translates into the requirement that

$$\sum_{i=1}^{n} (Q_i - p) = 0 \,, \tag{4.4}$$

where one uses the natural addition of points on E discussed above.

Thus, on a given elliptic curve, giving a semi-stable $\mathrm{U}(n)$ bundle is equivalent to giving an unordered (because of the Weyl symmetry) n-tuple of points on the curve. An $\mathrm{SU}(n)$ bundle has the further restriction that $\sum_i (Q_i - p) = 0$. For an $\mathrm{SU}(n)$ bundle, these points can be represented very explicitly as roots of an equation in the Weierstrass coordinates describing the elliptic curve. In affine coordinates, where $z = 1$, we write

$$s = a_0 + a_2 x + a_3 y + a_4 x^2 + a_5 x^2 y + \cdots + a_n x^{n/2}. \tag{4.5}$$

(If n is odd the last term is $a_n x^{(n-3)/2} y$.) Solving the equation $s = 0$, together with the Weierstrass equation (hence the appearance of only linear terms in y in s), gives n roots corresponding to the n points Q_i, where one can show that $\sum_i (Q_i - p) = 0$ as required. One notes that the roots are determined by the coefficients a_i only up to an overall scale factor. Thus the moduli space of roots Q_i is the projective space $\mathbb{P}^{n-1}$ as anticipated, with the coefficients a_i acting as homogeneous coordinates.

In summary, semi-stable $\mathrm{U}(n)$ bundles on an elliptic curve are described by an unordered n-tuple of points Q_i on the elliptic curve. $\mathrm{SU}(n)$ bundles have the additional condition that $\sum_i (Q_i - p) = 0$. In the $\mathrm{SU}(n)$ case, these points can be realized as roots of the equation $s = 0$ and give a moduli space of bundles which is simply $\mathbb{P}^{n-1}$, as mentioned above.

4.2 The spectral cover and the line bundle $\mathcal{N}$

Given that a bundle on an elliptic curve is described by the n-tuple Q_i, it seems reasonable that a bundle on an elliptic fibration determines how the n points vary as one moves around the base B. The set of all the n points over the base is called the spectral cover C and is an n-fold cover of B with $\pi_C : C \to B$. The spectral cover alone does not contain enough information to allow us to construct the bundle V. To do this, one must specify an additional line bundle, denoted by $\mathcal{N}$, on the spectral cover C. One obtains $\mathcal{N}$, given the vector bundle V, as follows. Consider the elliptic fiber E_b at any point $b \in B$. It follows from the previous section that

$$V, E_b| = \mathcal{N}_{1b} \oplus \cdots \oplus \mathcal{N}_{nb}\,, \tag{4.6}$$

where $\mathcal{N}_{ib}$ for $i = 1, \ldots, n$ are line bundles on E_b. In particular, we get a decomposition of the fiber $V_{\sigma(b)}$ of V at $p = \sigma(b)$. Let $V|_B$ be the restriction of V to the base B embedded in X via the section σ. We have just shown that the n-dimensional fibers of $V|_B$ come equipped with a decomposition into a sum of lines. As point b moves around the base B, these n lines move in one to one correspondence with the n points Q_i above b. This data specifies a unique line bundle[1] $\mathcal{N}$ on C such that the direct image $\pi_{C*}\mathcal{N}$ is $V|_B$ with its given decomposition. The direct image $\pi_{C*}\mathcal{N}$ is a vector bundle on B whose fiber at a generic point b, where the inverse image $\pi_C^{-1}(b)$ consists of the n distinct points Q_i, is the direct sum of the n lines $\mathcal{N}|_{Q_i}$.

4.3 Construction of bundles

We are now in a a position to construct the rank n vector bundle starting with the spectral data [37, 38, 39]. The spectral data consists of the spectral cover $C \subset X$ together with the line bundle $\mathcal{N}$ on C. The spectral cover is a divisor (hypersurface) $C \subset X$ which is of degree n over the base B; that is, the restriction $\pi_C : C \to B$ of the elliptic fibration is an n-sheeted branched cover. Equivalently, the cohomology class of C in $H^2(X, \mathbb{Z})$ must be of the form

$$[C] = n\sigma + \eta\,, \tag{4.7}$$

where η is a class in $H^2(B, \mathbb{Z})$ and σ is the section. This is equivalent to saying that the line bundle $\mathcal{O}_X(C)$ on X determined by C, whose sections are meromorphic functions on X with simple poles along C, is given by

$$\mathcal{O}_X(C) = \mathcal{O}_X(n\sigma) \otimes \mathcal{M}\,, \tag{4.8}$$

where $\mathcal{M}$ is some line bundle on X whose restriction to each fiber E_b is of degree zero. Written in this formulation

$$\eta = c_1(\mathcal{M}). \tag{4.9}$$

The line bundle $\mathcal{N}$ is, at this point, completely arbitrary.

[1]When C is singular, $\mathcal{N}$ may be more generally a rank-1 torsion free sheaf on C. For non-singular C this is the same as a line bundle.

Given this data, one can construct a rank n vector bundle V on X. It is easy to describe the restriction $V|_B$ of V to the base B. It is simply the direct image $V|_B = \pi_{C*}\mathcal{N}$. It is also easy to describe the restriction of V to a general elliptic fiber E_b. Let $C \cap E_b = \pi_C^{-1}(b) = Q_1 + \cdots + Q_n$ and $\sigma \cap E_b = p$. Then each Q_i determines a line bundle $\mathcal{N}_i$ of degree zero on E_b whose sections are the meromorphic functions on E_b with first order poles at Q_i which vanish at p. The restriction $V|_{E_b}$ is then the sum of the $\mathcal{N}_i$. Now the main point is that there is a unique vector bundle V on X with these specified restrictions to the base and the fibers.

To describe the entire vector bundle V, we use the Poincare bundle $\mathcal{P}$. This is a line bundle on the fiber product $X \times_B X'$. Here X' is the "dual fibration" to X. In general, this is another elliptic fibration which is locally, but not globally, isomorphic to X. However, when X has a section (which we assume), then X and X' are globally isomorphic, so we can identify them if we wish. (Actually, the spectral cover C lives most naturally as a hypersurface in the dual X', not in X. When we described it above as living in X, we were implicitly using the identification of X and X'.) The fiber product $X \times_B X'$ is a space of complex dimension four. It is fibered over B, the fiber over $b \in B$ being the ordinary product $E_b \times E'_b$ of the two fibers. Now, the Poincare bundle $\mathcal{P}$ is determined by the following two properties: (1) its restriction $\mathcal{P}|_{E_b \times x}$ to a fiber $E_b \times x$, for $x \in E'_b$, is the line bundle on E_b determined by x while (2) its restriction to $\sigma \times_B X'$ is the trivial bundle. Explicitly, $\mathcal{P}$ can be given by the bundle whose sections are meromorphic functions on $X \times_B X'$ with first order poles on $\mathcal{D}$ and which vanish on $\sigma \times_B X'$ and on $X \times_B \sigma'$. That is

$$\mathcal{P} = \mathcal{O}_{X \times_B X'}(\mathcal{D} - \sigma \times_B X' - X \times_B \sigma') \otimes K_B\,, \tag{4.10}$$

where $\mathcal{D}$ is the diagonal divisor representing the graph of the isomorphism $X \to X'$.

Using this Poincare bundle, we can finally describe the entire vector bundle V in terms of the spectral data. It is given by

$$V = \pi_{1*}(\pi_2{}^*\mathcal{N} \otimes \mathcal{P})\,. \tag{4.11}$$

Here π_1 and π_2 are the two projections of the fiber product $X \times_B C$ onto the two factors X and C. The two properties of the Poincare bundle guarantee that the restrictions of this V to the base and the fibers indeed agree with the intuitive versions of V_B and $V|_{E_b}$ given above.

In general, this procedure produces $\mathrm{U}(n)$ bundles. In order to get $\mathrm{SU}(n)$ bundles, two additional conditions must hold. First, the condition that the line bundle $\mathcal{M}$ in eq. (4.8) has degree zero on each fiber E_b must be strengthened to require that the restriction of $\mathcal{M}$ to E_b is the trivial bundle. Hence, $\mathcal{M}$ is the pullback to X of a line bundle on B which, for simplicity, we also denote by $\mathcal{M}$. This guarantees that the restrictions to the fibers $V|_{E_b}$ are $\mathrm{SU}(n)$ bundles. The second condition is that $V|_B$ must be an $\mathrm{SU}(n)$ bundle as well. That is, the line bundle $\mathcal{N}$ on C is such that the first Chern class c_1 of the resulting bundle V vanishes. This condition, and its ramifications, will be discussed in the next section.

U(n) vector bundles on the orbifold planes of heterotic M-theory are always sub-bundles of an E_8 vector bundle. As such, issues arise concerning their stability or semi-stability which are important and require considerable analysis. Furthermore, the associated Chern classes require an extended analysis to compute. For these reasons, in this paper, we will limit our discussion to SU(n) bundles, which are easier to study, and postpone the important discussion of U(n) bundles for a future publication [53].

4.4 Chern classes and restrictions on the bundle

As discussed above, the global condition that the bundle be SU(n) is that

$$c_1(V) = (1/2\pi)\mathrm{tr}\, F = 0\,. \tag{4.12}$$

This condition is clearly true since, for structure group SU(n), the trace must vanish. A formula for $c_1(V)$ can be extracted from the discussion in Friedman, Morgan and Witten [37]. One finds that

$$c_1(V) = \pi_{C*}\left(c_1(\mathcal{N}) + \frac{1}{2}\,c_1(C) - \frac{1}{2}\,\pi^* c_1(B)\right)\,, \tag{4.13}$$

where $c_1(B)$ means the first Chern class of the tangent bundle of B considered as a complex vector bundle, and similarly for C, while π_C is the projection from the spectral cover onto B; that is, $\pi_C : C \to B$. The operators π_C^* and π_{C*} are the pull-back and push-forward of cohomology classes between B and C. The condition that $c_1(V)$ is zero then implies that

$$c_1(\mathcal{N}) = -\frac{1}{2}\,c_1(C) + \frac{1}{2}\,\pi_C^* c_1(B) + \gamma\,, \tag{4.14}$$

where γ is some cohomology class satisfying the equation

$$\pi_{C*}\gamma = 0\,. \tag{4.15}$$

The general solution for γ constructed from cohomology classes is

$$\gamma = \lambda\big(n\sigma - \pi_C^*\eta + n\pi_C^* c_1(B)\big)\,, \tag{4.16}$$

where λ is a rational number and σ is the global section of the elliptic fibration. Appropriate values for λ will emerge shortly. From (4.7) we recall that $c_1(C)$ is given by

$$c_1(C) = -n\sigma - \pi_C^*\eta\,. \tag{4.17}$$

Combining the eqs. (4.14), (4.16), and (4.17) yields

$$c_1(\mathcal{N}) = n\left(\frac{1}{2}+\lambda\right)\sigma + \left(\frac{1}{2}-\lambda\right)\pi_C^*\eta + \left(\frac{1}{2}+n\lambda\right)\pi_C^* c_1(B)\,. \tag{4.18}$$

Essentially, this means that the bundle $\mathcal{N}$ is completely determined in terms of the elliptic fibration and $\mathcal{M}$. It is important to note, however, that there is not always a solution for $\mathcal{N}$. The reason for this is that $c_1(\mathcal{N})$ must be integer, a condition that puts a substantial constraint on the allowed bundles. To see this, note that the section is a horizontal divisor, having unit intersection number with the elliptic fiber. On the other hand, the quantities $\pi_C^* c_1(B)$ and $\pi_C^*\eta$ are vertical, corresponding to curves in the base lifted to the fiber and so have zero intersection number with the fiber. Therefore, we cannot choose η to cancel σ and, hence, the coefficient of σ must, by itself, be an integer . This implies that a consistent bundle $\mathcal{N}$ will exist if either

$$n \text{ is odd}\,, \quad \lambda = m + \frac{1}{2} \tag{4.19}$$

or

$$n \text{ is even}\,, \quad \lambda = m\,, \qquad \eta = c_1(B) \bmod 2\,, \tag{4.20}$$

where m is an integer. Here, the $\eta = c_1(B) \bmod 2$ condition means that η and $c_1(B)$ differ by an even element of $H^2(B,\mathbb{Z})$. Note that when n is even, we cannot choose η arbitrarily. These conditions are only sufficient for the existence of a consistent line bundle $\mathcal{N}$. They are also sufficient for all the examples we consider in this paper, and are the only classes of solutions which is easy to describe in general. However, other solutions do exist. We could, for example take $n = 4$, $\lambda = (1/4)$, and $\eta = 2c_1(B)$ (mod 4), or $n = 5$, $\lambda = (1/10)$, and $\eta = 0$ (mod 5).

Finally, we can give the explicit Chern classes for the SU(n) vector bundle V. Friedman, Morgan and Witten calculate $c_1(V)$ and $c_2(V)$, while Curio and Andreas [44, 45] have found $c_3(V)$. The results are

$$c_1(V) = 0\,, \tag{4.21}$$

$$c_2(V) = \eta\sigma - \frac{1}{24}c_1(B)^2\left(n^3-n\right) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta\left(\eta - nc_1(B)\right), \tag{4.22}$$

$$c_3(V) = 2\lambda\sigma\eta\left(\eta - nc_1(B)\right), \tag{4.23}$$

where the wedge product is understood.

5. Summary of elliptic fibrations and bundles

The previous two sections are somewhat abstract. For the sake of clarity, we will here summarize those results which are directly relevant to constructing physically acceptable non-perturbative vacua. First consider the Calabi-Yau space.

- An elliptically fibered Calabi-Yau three-fold is composed of a two-fold base B and elliptic curves E_b fibered over each point $b \in B$. In this paper, we consider only those elliptic fibrations that admit a global section σ.

- The elliptic fibration is characterized by a single line bundle $\mathcal{L}$ over B. The vanishing of the first Chern class of the canonical bundle K_X of the Calabi-Yau three-fold X implies that

$$\mathcal{L} = K_B^{-1}\,, \tag{5.1}$$

  where K_B is the canonical bundle of the base B.

- From the previous condition, it follows that the base B is restricted to del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces.

- The second Chern class of the holomorphic tangent bundle of X is given by

$$c_2(TX) = c_2(B) + 11c_1(B)^2 + 12\sigma c_1(B)\,, \tag{5.2}$$

  where $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of B.

Next we summarize the spectral cover construction of semi-stable holomorphic gauge bundles.

- A general semi-stable $\mathrm{SU}(n)$ gauge bundle V is determined by two line bundles, $\mathcal{M}$ and $\mathcal{N}$. The relevant quantities associated with $\mathcal{M}$ and $\mathcal{N}$ are their first Chern classes

$$\eta = c_1(\mathcal{M}) \tag{5.3}$$

  and $c_1(\mathcal{N})$ respectively. The class $c_1(\mathcal{N})$, in addition to depending on n, σ, $c_1(B)$ and η, also contains a rational number λ.

- The condition that $c_1(\mathcal{N})$ be an integer leads to the sufficient but not necessary constraints on η and λ given by

$$\begin{aligned} &n \text{ is odd}\,, \quad \lambda = m + \frac{1}{2}\,, \\ &n \text{ is even}\,, \quad \lambda = m\,, \qquad \eta = c_1(B) \bmod 2\,, \end{aligned} \tag{5.4}$$

  where m is an integer.

- The relevant Chern classes of an $\mathrm{SU}(n)$ gauge bundle V are given by

$$c_1(V) = 0\,, \tag{5.5a}$$

$$c_2(V) = \eta\sigma - \frac{1}{24}c_1(B)^2\left(n^3 - n\right) + \frac{1}{2}\left(\lambda^2 - \frac{1}{4}\right)n\eta\big(\eta - nc_1(B)\big), \tag{5.5b}$$

$$c_3(V) = 2\lambda\sigma\eta\big(\eta - nc_1(B)\big). \tag{5.5c}$$

How can one use the this data to construct realistic particle physics theories? One proceeds as follows.

- Choose a base B from one of the allowed bases; namely, a del Pezzo, Hirzebruch or Enriques surface, or a blow-up of a Hirzebruch surface. The associated Chern classes $c_1(B)$ and $c_2(B)$ can be computed for any of these surfaces.

This allows one to construct the second Chern class of the Calabi-Yau tangent bundle and a part of the gauge bundle Chern classes.

- Specify η and λ subject to the constraints (5.4).

Given appropriate η and λ, one can completely determine the relevant gauge bundle Chern classes. In addition we must satisfy the cohomology condition (2.10). This relates the Chern classes to the curves on which the five-branes wrap and it is what we will turn to next.

6. Effective curves and five-branes

Let us now consider the five-branes. Having discussed the Calabi-Yau three-fold X and the gauge bundle, the third ingredient for defining a non-perturbative vacuum, is to give a set of five-branes wrapped on holomorphic curves within X. We recall that the cohomology condition (2.10) fixes the cohomology class associated with the five-branes so that

$$[W] = c_2(TX) - c_2(V_1) - c_2(V_2)\,. \tag{6.1}$$

In order to make physical sense, this class must be Poincaré dual to the homology class of a set of curves in the Calabi-Yau space. As discussed in appendix A, this means that $[W]$ must be effective. In general, this will restrict the bundles V_i one can choose. To understand the form of this restriction we need to find the set of effective curves on X.

Consider a complex manifold X which is an elliptic fibration over a base B. Let us suppose we have found an effective class in $H_2(B,\mathbb{Z})$. Then, it naturally also lies in an effective homology class in $H_2(X,\mathbb{Z})$ of the elliptic fibration. Note that the fibration structure guarantees that if two curves are in different classes in the base, then they are in different classes in the full manifold X. This implies, among other things, that if one finds the effective generating class of the Mori cone of B, these classes remain distinct classes of X. In addition, there is at least one other effective class that is not associated with the base. This is the class F of the fiber itself. There may also be other effective classes, for example, those related to points where the fiber degenerates. However, we will ignore these since they will not appear in the homology classes of the five-branes, our main interest in this paper.

The algebraic classes that do appear naturally are quadratic polynomials in classes of the line bundles. The line bundle classes are represented by two-forms or equivalently, under Poincaré duality, by divisors, which are surfaces in X. Classes such as $c_2(V_i)$, given in (5.5b), are quadratic in these line bundle classes. In terms of homology they are represented by curves corresponding to the intersection of two divisors. The only line bundle classes on a general elliptically fibered Calabi-Yau three-fold X are the base B and the divisors $\pi^{-1}(\mathcal{C})$, where $\mathcal{C}$ is a curve in B. Any quadratic polynomial in these classes can be written as

$$W = W_B + a_f F\,, \tag{6.2}$$

where W_B is an algebraic homology class in the base manifold B embedded in X and a_f is some integer. Under what conditions is W an effective class? It is clear that W is effective if W_B is an effective class in the base and $a_f \geq 0$. One can also prove that the converse is true in almost all cases. One sees this as follows. First, unless a curve is purely in the fiber, in which case $W_B = 0$, the fact that X is elliptically fibered means that all curves W project to curves in the base. The class $[W]$ similarly projects to the class W_B. The projection of an effective class must be effective, thus if $[W]$ is effective in X then so is W_B in B. The only question then is whether there are effective curves in X with negative a_f. To address this we use the fact that any effective curve must have non-negative intersection with any effective divisor in X unless the curve is contained within the divisor. The intersections of $[W]$ with the effective divisor classes mentioned above are given by

$$\pi_* \left(\pi^{-1}(\mathcal{C}) \cdot W\right) = \mathcal{C} \cdot W_B\,, \qquad \pi_* \left(B \cdot W\right) = K_B \cdot W_B + a_f\,, \tag{6.3}$$

where the intersections on the right-hand side are for classes in the base. The second expression is derived by adjunction, recalling, from (3.9), that the normal bundle to B is $\mathcal{N}_{B/X} = \mathcal{L} = -K_B$. From the first intersection one simply deduces again that if $[W]$ is effective then so is W_B. Suppose that a_f is nonzero. Then W cannot be contained within B and so from the second expression we have $a_f \geq -K_B \cdot W_B$. From appendix B we recall that for del Pezzo and Enriques surfaces, $-K_B$ is nef, so that its intersection with any effective class W_B is nonnegative. Thus we do have $a_f \geq 0$ for $[W]$ to be effective. The exception is a Hirzebruch surface for $r \geq 3$. We then have $-K_B \cdot \mathcal{E} = 2 > 0$ but $-K_B \cdot S = 2 - r < 0$.

In conclusion, we see that, first $[W]$ is effective if and only if W_B is an effective class in B and $a_f \geq 0$ for any del Pezzo or Enriques surface. Second, this is also true for a Hirzebruch surface F_r, with the exception of when W_B happens to contain the negative section S and $r \geq 3$. In this paper, for simplicity, we will consider only those cases for which the statement is true. Thus, under this restriction, we have that

$$W \text{ is effective } \Longleftrightarrow W_B \text{ is effective in } B \text{ and } a_f \geq 0\,. \tag{6.4}$$

This reduces the question of finding the effective curves in X to knowing the generating set of effective curves in the base B. For the set of base surfaces B we are considering, finding such generators is always possible.

For simplicity, in this paper we will allow for arbitrary semi-stable gauge bundles V_1, which we henceforth call V, on the first orbifold plane, but always take the gauge bundle V_2 to be trivial. Physically, this corresponds to allowing observable sector gauge groups to be subgroups, such as SU(5), SO(10), or E_6, of E_8 but leaving the hidden sector E_8 gauge group unbroken. We do this only for simplicity. Our formalism also allows an analysis of the general case where the hidden sector E_8 gauge group is broken by a non-trivial bundle V_2. With this restriction, eq. (6.1) simplifies to

$$[W] = c_2(TX) - c_2(V). \tag{6.5}$$

Inserting the expressions (5.2) and (4.22) for the second Chern classes, we find that

$$[W] = W_B + a_f F\,, \tag{6.6}$$

where

$$W_B = \sigma\big(12c_1(B) - \eta\big) \tag{6.7}$$

is the part of the class associated with the base B and

$$a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right)c_1(B)^2 - \frac{1}{2}\,n\left(\lambda^2 - \frac{1}{4}\right)\eta\big(\eta - nc_1(B)\big). \tag{6.8}$$

is the part associated with the elliptic fiber.

As we have already stated, to make physical sense, $[W]$ must be an effective class. This physical requirement then implies, using the theorem (6.4), that necessarily

$$W_B \text{ is effective in } B, \qquad a_f \geq 0\,. \tag{6.9}$$

This puts a further constraint on the allowed bundles in non-perturbative vacua. Note, however, that this condition is much weaker that the corresponding constraint without five-branes. In that case W_B and a_f must vanish. It is this additional freedom which greatly facilitates the construction of suitable particle physics vacua.

7. Number of families and model building rules

The first obvious physical criterion for constructing realistic particle physics models is that we should be able to find theories with a small number of families, preferably three. We will see that this is, in fact, easy to do via the bundle constructions on elliptically fibered Calabi-Yau three-folds that we are discussing. We start by deriving the three family criterion as discussed, for instance, in Green, Schwarz, and Witten [56]. The form of this condition for elliptically fibered Calabi-Yau manifolds was first given by Curio [44].

The number of families is related to the number of zero-modes of the Dirac operator in the presence of the gauge bundle on the Calabi-Yau three-fold, since we want to count the number of massless fermions of different chiralities. The original gauginos are in the adjoint representation of E_8. In this paper, we are considering only gauge bundles V with $\mathrm{SU}(n)$ fiber groups. To count the number of families, we need to count the number of fields in the matter representations of the low energy gauge group, that is, the subgroup of E_8 commutant with $\mathrm{SU}(n)$, and their complex conjugates respectively. Explicitly, in this paper, we will be interested in the following breaking patterns

$$\begin{aligned}
E_8 \supset \mathrm{SU}(3) \times E_6 &: \mathbf{248} = (\mathbf{8},\mathbf{1}) \oplus (\mathbf{1},\mathbf{78}) \oplus (\mathbf{3},\mathbf{27}) \oplus (\bar{\mathbf{3}},\bar{\mathbf{27}}), \\
E_8 \supset \mathrm{SU}(4) \times \mathrm{SO}(10) &: \mathbf{248} = (\mathbf{15},\mathbf{1}) \oplus (\mathbf{1},\mathbf{45}) \oplus (\mathbf{4},\mathbf{16}) \oplus (\bar{\mathbf{4}},\bar{\mathbf{16}}) \oplus (\mathbf{6},\mathbf{10}), \\
E_8 \supset \mathrm{SU}(5) \times \mathrm{SU}(5) &: \mathbf{248} = (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus (\bar{\mathbf{10}},\bar{\mathbf{5}}) \oplus \\
&\qquad \oplus (\mathbf{5},\bar{\mathbf{10}}) \oplus (\bar{\mathbf{5}},\mathbf{10}).
\end{aligned} \tag{7.1}$$

Note, however, that the methods presented here will apply to any breaking pattern with an $\mathrm{SU}(n)$ subgroup. We see that all the matter representations appear in the fundamental representation of the structure group $\mathrm{SU}(n)$. By definition, the index of the Dirac operator measures the difference in the number of positive and negative chirality spinors, in this case, on the Calabi-Yau three-fold. Since six-dimensional chirality is correlated with four-dimensional chirality, the index gives the number of families. From the fact that all the relevant fields are in the fundamental representation of $\mathrm{SU}(n)$, we have that the number of generations is

$$N_{\mathrm{gen}} = \mathrm{index}(V, \not{D}) = \int_X \mathrm{td}(X)\,\mathrm{ch}(V) = \frac{1}{2}\int_X c_3(V)\,, \tag{7.2}$$

where $\mathrm{td}(X)$ is the Todd class of X. For the case of $\mathrm{SU}(n)$ bundles on elliptically fibered Calabi-Yau three-folds, one can show, using eq. (4.23) above, that the number of families becomes

$$N_{\mathrm{gen}} = \lambda\eta\big(\eta - nc_1(B)\big)\,, \tag{7.3}$$

where we have integrated over the fiber. Hence, to obtain three families the bundle must be constrained so that

$$3 = \lambda\eta\big(\eta - nc_1(B)\big). \tag{7.4}$$

It is useful to express this condition in terms of the class W_B given in eq. (6.7) and integrated over the fiber. We find that

$$3 = \lambda\left(W_B^2 - (24-n)W_Bc_1(B) + 12(12-n)c_1(B)^2\right). \tag{7.5}$$

Furthermore, inserting the three family constraint into (6.8) gives

$$a_f = c_2(B) + \left(11 + \frac{1}{24}(n^3 - n)\right)c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right). \tag{7.6}$$

We are now in a position to summarize all the rules and constraints that are required to produce particle physics theories with three families. We have that the homology class associated with the five-branes is specifically of the form

$$[W] = W_B + a_f F\,, \tag{7.7}$$

where

$$W_B = \sigma(12c_1(B) - \eta), \tag{7.8}$$

$$a_f = c_2(B) + \left(11 + \frac{1}{24}(n^3 - n)\right)c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) \tag{7.9}$$

and $c_1(B)$ and $c_2(B)$ are the first and second Chern classes of B.

The constraints for constructing particle physics vacua are then

- Effective condition: The requirement that $[W]$ is the class of a set of physical five-branes constrains $[W]$ to be an effective. Therefore, we must guarantee that

$$W_B \text{ is effective in } B\,, \quad a_f \geq 0 \text{ integer} \tag{7.10}$$

- Three-family condition: The requirement that the theory have three families imposes the further constraint that

$$3 = \lambda\left(W_B^2 - (24 - n)W_B c_1(B) + 12(12 - n)c_1(B)^2\right). \tag{7.11}$$

To these conditions, we can add the remaining relevant constraint from section 4. It is

- Bundle condition: The condition that $c_1(\mathcal{N})$ be an integer leads to the constraints on W_B and λ given by

$$\begin{aligned} &n \text{ is odd}\,, \quad \lambda = m + \frac{1}{2}\,, \\ &n \text{ is even}\,, \quad \lambda = m\,, \qquad W_B = c_1(B) \bmod 2\,, \end{aligned} \tag{7.12}$$

where m is an integer. Recall that this condition is sufficient, but not necessary.

Note that in this last condition, the class η, which appeared in constraint (5.4), has been replaced by W_B. That this replacement is valid can be seen as follows. For n odd, there is no constraint on η and, hence, using (7.8), no constraint on W_B. When n is even, it is sufficient for η to satisfy $\eta = c_1(B) \bmod 2$. Since $12c_1(B)$ is an even element of $H^2(B, \mathbb{Z})$, it follows that $W_B = c_1(B) \bmod 2$.

It is important to note that all quantities and constraints have now been reduced to properties of the base two-fold B. Specifically, if we know $c_1(B)$, $c_2(B)$, as well as a set of generators of effective classes in B in which to expand W_B, we will be able to exactly specify all appropriate non-perturbative vacua. For the del Pezzo, Hirzebruch, Enriques, and blown-up Hirzebruch surfaces, all of these quantities are known.

Finally, from the expressions in (7.1) we find the following rule.

- If we denote by G the structure group of the gauge bundle and by H its commutant subgroup, then

$$G = \mathrm{SU}(3) \Longrightarrow H = E_6\,, \tag{7.13}$$
$$G = \mathrm{SU}(4) \Longrightarrow H = \mathrm{SO}(10)\,, \tag{7.14}$$
$$G = \mathrm{SU}(5) \Longrightarrow H = \mathrm{SU}(5)\,, \tag{7.15}$$

  H corresponds to the low energy gauge group of the theory.

Armed with the above rules, we now turn to the explicit construction of phenomenologically relevant non-perturbative vacua.

8. Three family models

In this section, we will construct four explicit solutions satisfying the above rules. In general, we will look for solutions where the class representing the curve on which the fivebranes wrap is comparatively simple. As discussed above, the allowed base surfaces B of elliptically fibered Calabi-Yau three-folds which admit a section are restricted to be the del Pezzo, Hirzebruch and Enriques surfaces, as well as blow-ups of Hirzebruch surfaces. Relevant properties of del Pezzo, Hirzebruch and Enriques surfaces, including their generators of effective curves, are given in the appendix B. However, we now show that Calabi-Yau three-folds of this type with an Enriques base never admit an effective five-brane curve if one requires that there be three families. Recall that the cohomology class of the spectral cover must be of the form

$$[C] = n\sigma + \eta \tag{8.1}$$

and this necessarily is an effective class in X. We may assume that C does not contain $\sigma(B)$. Otherwise, replace C in the following discussion with its subcover C'

obtained by discarding the appropriate multiples of $\sigma(B)$. This implies that the class of the intersection of σ with $[C]$

$$\sigma[C] = n\sigma^2 + \sigma\eta \tag{8.2}$$

must be effective in the base B. Let us restrict B to be an Enriques surface. Using the adjunction formula, we find that

$$\sigma^2 = K_B\,, \tag{8.3}$$

where K_B is the torsion class. Since nK_B vanishes for even n, it follows that when n is even

$$\sigma[C] = \sigma\eta\,. \tag{8.4}$$

Clearly, $\sigma\eta$ is effective, since $\sigma[C]$ is. For n odd, $nK_B = K_B$ and, hence

$$\sigma[C] = K_B + \sigma\eta\,. \tag{8.5}$$

Using the discussion in appendix B, one can still conclude that $\sigma\eta$ is either an effective class or it equals K_B. From the fact that

$$\sigma c_1(B) = K_B \tag{8.6}$$

it follows, using eq. (7.8), that the five-brane class restricted to the Enriques base is given by

$$W_B = 12K_B - \sigma\eta\,. \tag{8.7}$$

Since $12K_B$ vanishes, this becomes

$$W_B = -\sigma\eta \tag{8.8}$$

from which we can conclude that W_B is never effective for non-vanishing class $\sigma\eta$. Since, as explained above, W_B must be effective for the five-branes to be physical, such theories must be discarded. The only possible loop-hole is when $\sigma\eta$ vanishes or equals K_B. However, in this case, it follows from (7.3) that

$$N_{\text{gen}} = 0 \tag{8.9}$$

which is also physically unacceptable. We conclude that, on general grounds, Calabi-Yau three-folds with an Enriques base never admit effective five-brane curves if one requires that there be three families.[2] For this reason, we henceforth restrict our discussion to the remaining possibilities. In this section, for specificity, the base B will always be chosen to be either a del Pezzo surface or a Hirzebruch surface.

We first give two SU(5) examples, each on del Pezzo surfaces; one where the base component, W_B, is simple and one where the fiber component has a small coefficient.

[2]We thank E. Witten for pointing out to us the likelihood of this conclusion.

Example 1: $B = dP_8$, $H = \mathrm{SU}(5)$. We begin by choosing

$$H = \mathrm{SU}(5) \tag{8.10}$$

as the gauge group for our model. Then it follows from (7.14) that we must choose the structure group of the gauge bundle to be

$$G = \mathrm{SU}(5) \tag{8.11}$$

and, hence, $n = 5$.

At this point, it is necessary to explicitly choose the base surface, which we take to be

$$B = dP_8 \,. \tag{8.12}$$

It follows from appendix B that for the del Pezzo surface dP_8, a basis for $H_2(dP_8, \mathbb{Z})$ composed entirely of effective classes is given by l and E_i for $i = 1, \ldots, 8$ where

$$l \cdot l = 1 \,, \qquad l \cdot E_i = 0 \,, \qquad E_i \cdot E_j = -\delta_{ij} \,. \tag{8.13}$$

There are other effective classes in dP_8 not obtainable as a linear combination of l and E_i with non-negative integer coefficients, but we will not need them in this example. To these we add the fiber class F. Furthermore

$$c_1(B) = 3l - \sum_{r=1}^{8} E_i \tag{8.14}$$

and

$$c_2(B) = 11 \,. \tag{8.15}$$

We now must specify the component of the five-brane class in the base and the coefficient λ subject to the three constraints (7.10), (7.11), and (7.12). Since n is odd, the bundle constraint (7.12) tells us that $\lambda = m + (1/2)$ for integer m. Here we will choose $m = 1$ and W_B such that

$$W_B = 2E_1 + E_2 + E_3 \,, \qquad \lambda = \frac{3}{2} \,. \tag{8.16}$$

Since E_1, E_2, and E_3 are effective, it follows that W_B is also effective, as it must be. Using the above intersection rules, one can easily show that

$$W_B^2 = -6 \,, \qquad W_B c_1(B) = 4 \,, \qquad c_1(B)^2 = 1 \,. \tag{8.17}$$

Using these results, as well as $n = 5$ and $\lambda = (3/2)$, one finds that

$$a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right) c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 17 \,. \tag{8.18}$$

Since this is a positive integer, we have satisfied the effectiveness condition (7.10) and the full five-brane class $[W]$ is effective in the Calabi-Yau three-fold X. Finally, we find that

$$\lambda\big(W_B^2 - (24-n)W_B c_1(B) + 12(12-n)c_1(B)^2\big) = 3 \tag{8.19}$$

and, therefore, the three family condition (7.11) is satisfied.

This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = \mathrm{SU}(5)$, along with explicit five-branes wrapped on a holomorphic curve with homology class

$$[W] = 2E_1 + E_2 + E_3 + 17F\,. \tag{8.20}$$

The properties of the moduli space of the five-branes were discussed in [36] and will be explored in more detail in a future publication [47].

Example 2: $B = dP_9$, $H = \mathrm{SU}(5)$. As a second example, we again choose gauge group

$$H = \mathrm{SU}(5) \tag{8.21}$$

and, hence, the structure group

$$G = \mathrm{SU}(5). \tag{8.22}$$

Then $n = 5$ and we can again choose $m = 1$ and, therefore

$$\lambda = \frac{3}{2}\,. \tag{8.23}$$

In this example, we will take as a base surface

$$B = dP_9\,. \tag{8.24}$$

It follows from appendix B that a basis for $H_2(dP_9, \mathbb{Z})$ composed entirely of effective classes is given by l and E_i for $i = 1, \ldots, 9$. In addition, there are other effective classes in dP_9 not obtainable as linear combinations of l and E_i with non-negative integer coefficients. One such effective class is

$$c_1(B) = 3l - \sum_{r=1}^{9} E_i\,. \tag{8.25}$$

Furthermore

$$c_2(B) = 12\,. \tag{8.26}$$

We now must specify the component of the five-brane class in the base. In this example, we choose

$$W_B = 6E_1 + E_2 + E_3 + 12\left(3l - \sum_{i=1}^{9} E_i\right). \tag{8.27}$$

Since E_1, E_2, E_3, and $3l - \sum_{i=1}^{9} E_i$ are effective, it follows that W_B is also effective, as it must be. Using the above intersection rules, one can easily show that

$$W_B^2 = 154\,, \qquad W_B c_1(B) = 8\,, \qquad c_1(B)^2 = 0\,. \tag{8.28}$$

Using these results, as well as $n = 5$ and $\lambda = (3/2)$ one can check that

$$\lambda\big(W_B^2 - (24-n)W_B c_1(B) + 12(12-n)c_1(B)^2\big) = 3 \tag{8.29}$$

and, therefore, the three family condition is satisfied. Finally, let us compute the coefficient a_f of F. Using the above information, we find that

$$a_f = c_2(B) + \left(11 + \frac{n^3-n}{24}\right)c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 2\,. \tag{8.30}$$

Since this is a positive integer, it follows from the above discussion that the full five-brane curve $[W]$ is effective in the Calabi-Yau three-fold, as it must be. This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = \mathrm{SU}(5)$, along with explicit five-branes wrapped on a holomorphic curve specified by

$$[W] = 6E_1 + E_2 + E_3 + 12\left(3l - \sum_{i=1}^{9} E_i\right) + 2F\,. \tag{8.31}$$

Still within the context of del Pezzo base manifolds, we now give a third example, this time with gauge group $H = \mathrm{SO}(10)$.

Example 3: $B = dP_8$, $H = \mathrm{SO}(10)$. In this third example, we choose the gauge group to be

$$H = \mathrm{SO}(10) \tag{8.32}$$

and, hence, the structure group

$$G = \mathrm{SU}(4). \tag{8.33}$$

Then $n = 4$. Since n is even, then from constraint (7.12) we must have $\lambda = m$ where m is an integer and $W_B = c_1(B) \bmod 2$. Here we will choose $m = -1$ so that

$$\lambda = -1\,. \tag{8.34}$$

We will return to the choice of W_B momentarily. In this example, we will take as a base surface

$$B = dP_8\,. \tag{8.35}$$

Some of the effective generators and the first and second Chern classes of dP_8 were given in the previous example. We now must specify the component of the five-brane class in the base. In this example, we choose

$$W_B = 2E_1 + 2E_2 + \left(3l - \sum_{i=1}^{8} E_i\right)\,. \tag{8.36}$$

Since E_1, E_2, and $3l - \sum_{i=1}^{8} E_i$ are effective, it follows that W_B is also effective, as it must be. Furthermore, since

$$c_1(B) = 3l - \sum_{r=1}^{8} E_i \tag{8.37}$$

it follows that

$$W_B = c_1(B) \bmod 2 \tag{8.38}$$

since $2E_1 + 2E_2$ is an even element of $H^2(dP_9, \mathbb{Z})$. Using the above intersection rules, one can easily show that

$$W_B^2 = 1\,, \qquad W_B c_1(B) = 5\,, \qquad c_1(B)^2 = 1\,. \tag{8.39}$$

Using these results, as well as $n = 4$ and $\lambda = -1$, one can check that

$$\lambda\left(W_B^2 - (24-n)W_B c_1(B) + 12(12-n)c_1(B)^2\right) = 3 \tag{8.40}$$

and, therefore, the three family condition is satisfied. Finally, let us compute the coefficient a_f of F. Using the above information, we find that

$$a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right) c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 29\,. \tag{8.41}$$

Since this is a positive integer, it follows from the above discussion that the full five-brane curve $[W]$ is effective, as it must be. This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = \mathrm{SO}(10)$, along with explicit five-branes wrapped on a holomorphic curve specified by

$$[W] = 2E_1 + 2E_2 + \left(3l - \sum_{i=1}^{8} E_i\right) + 29F\,. \tag{8.42}$$

Example 4: $B = F_r$, $H = \mathrm{SU}(5)$. We now return to choosing

$$H = \mathrm{SU}(5) \tag{8.43}$$

as the gauge group for our model. Then it follows from (7.14) that we must choose the structure group of the gauge bundle to be

$$G = \mathrm{SU}(5) \tag{8.44}$$

and, hence, $n = 5$. Since n is odd, constraint (7.12) tells us that $\lambda = m + (1/2)$ for integer m. Here we will take $m = 0$ and, therefore

$$\lambda = \frac{1}{2}\,. \tag{8.45}$$

In this example, we will choose the base surface to be a general Hirzebruch surface

$$B = F_r\,, \tag{8.46}$$

where r is any non-negative integer. It follows from appendix B that for the Hirzebruch surface F_r, a basis for $H_2(F_r, \mathbb{Z})$ composed entirely of effective classes is given by S and $\mathcal{E}$ where

$$\mathcal{E} \cdot \mathcal{E} = 0\,, \qquad S \cdot S = -r\,, \qquad S \cdot \mathcal{E} = 1\,. \tag{8.47}$$

Furthermore, these completely generate the set of all effective classes. To these classes we add the fiber class F. In addition

$$c_1(B) = 2S + (r+2)\mathcal{E} \tag{8.48}$$

and

$$c_2(B) = 4\,. \tag{8.49}$$

We now must specify the component of the five-brane class in the base. In this example, we choose

$$W_B = 26S + (13r + 23)\mathcal{E}\,. \tag{8.50}$$

Since S and $\mathcal{E}$ are effective, it follows that W_B is also effective, as it must be. Using the above intersection rules, one can easily show that

$$W_B^2 = 1196\,, \qquad W_B c_1(B) = 98\,, \qquad c_1(B)^2 = 8\,. \tag{8.51}$$

Note that the integer r has canceled out of these expressions. Using these results, as well as $n = 5$ and $\lambda = (1/2)$, one can check that

$$\lambda\big(W_B^2 - (24-n)W_B c_1(B) + 12(12-n)c_1(B)^2\big) = 3 \tag{8.52}$$

and, therefore, the three family condition is satisfied. Finally, let us compute the coefficient a_f of F. Using the above information, we find that

$$a_f = c_2(B) + \left(11 + \frac{n^3 - n}{24}\right) c_1(B)^2 - \frac{3n}{2\lambda}\left(\lambda^2 - \frac{1}{4}\right) = 132\,. \tag{8.53}$$

Since this is a positive integer, it follows from the above discussion that the full five-brane curve $[W]$ is effective, as it must be. This completes our construction of this explicit non-perturbative vacuum. It represents a model of particle physics with three families and gauge group $H = \mathrm{SU}(5)$, along with explicit five-branes wrapped on a holomorphic curve specified by

$$[W] = 26S + (13r + 23)\mathcal{E} + 132F\,. \tag{8.54}$$

To repeat, we will explore the properties of the moduli spaces of five-branes in detail in [47].

A. General concepts

A.1 Poincare duality and intersection numbers.

The relationship between the cohomology class $[W]$ of the five-branes and the holomorphic curves in X over which they are wrapped arises from the generic relationship between the cohomology and homology groups of a manifold. While this connection is a familiar one, since it is used extensively in this paper, we will give a brief description of it here.

Let $H_k(X, \mathbb{R})$ be the kth real homology group over an oriented manifold X. The elements of $H_k(X, \mathbb{R})$ are closed cycles of dimension k on X. Now, every element $\mathcal{C}_k$ of $H_k(X, \mathbb{R})$ can be considered as a linear functional on forms in the de Rham cohomology group $H^k_{\mathrm{DR}}(X, \mathbb{R})$ in the following way. Let ϕ be any element of $H^k_{\mathrm{DR}}(X, \mathbb{R})$. Then

$$\phi \to \int_{\mathcal{C}_k} \phi \tag{A.1}$$

defines a linear map from $H^k_{\mathrm{DR}}(X, \mathbb{R}) \to \mathbb{R}$ for any $\mathcal{C}_k$ in $H_k(X, \mathbb{R})$. This map is well defined since two cycles $\mathcal{C}_k$ give the same integral if they are equal in homology (since ϕ is closed). In fact, all such linear maps can be realized this way and so the homology group $H_k(X, \mathbb{R})$ is dual as a vector space to $H^k_{\mathrm{DR}}(X, \mathbb{R})$. This is simply the statement that homology and cohomology are dual to each other.

There is another notion of duality, Poincare duality, that must be discussed to complete the story. Let n be the real dimension of the manifold X. One is then familiar with the notion of Poincaré duality for forms, that the de Rham cohomology groups $H^k_{\rm DR}(X,\mathbb{R})$ and $H^{n-k}_{\rm DR}(X,\mathbb{R})$ are dual as vector spaces, as follows. Let ϕ be any element of $H^k_{\rm DR}(X,\mathbb{R})$. Then

$$\phi \to \int_X \phi \wedge \psi \tag{A.2}$$

defines a linear map from $H^k_{\rm DR}(X,\mathbb{R}) \to \mathbb{R}$ for any ψ in $H^{n-k}_{\rm DR}(X,\mathbb{R})$. Again all such maps can be realized this way, and $H^k_{\rm DR}(X,\mathbb{R})$ and $H^{n-k}_{\rm DR}(X,\mathbb{R})$ are dual vector spaces. Now, denote by $\eta_{\mathcal{C}_k}$ the element of $H^{n-k}_{\rm DR}(X,\mathbb{R})$ with the property that

$$\int_{\mathcal{C}_k} \phi = \int_X \phi \wedge \eta_{\mathcal{C}_k} \tag{A.3}$$

for all $\phi \in H^k_{\rm DR}(X,\mathbb{R})$. Then the mapping

$$\mathcal{C}_k \longrightarrow \eta_{\mathcal{C}_k} \tag{A.4}$$

defines an isomorphism between the homology group $H_k(X,\mathbb{R})$ and the cohomology group $H^{n-k}_{\rm DR}(X,\mathbb{R})$ since both are dual to $H^k_{\rm DR}(X,\mathbb{R})$. This is the final result we want, that is

$$H_k(X,\mathbb{R}) \cong H^{n-k}_{\rm DR}(X,\mathbb{R}). \tag{A.5}$$

For example, let X be a Calabi-Yau three-fold and $\mathcal{C}_2$ any homology two-cycle contained in $H_2(X,\mathbb{R})$. Then, by the above discussion, this two-dimensional cycle in X can be identified with a unique cohomology class $\eta_{\mathcal{C}_2}$ contained in $H^4_{\rm DR}(X,\mathbb{R})$, and vice versa. This expresses the exact relationship between the five-brane four-form $[W]$ and its associated holomorphic curve. We will refer to isomorphism (A.5) as the Poincare isomorphism and, loosely speaking, to the pair $\mathcal{C}_k$ and $\eta_{\mathcal{C}_k}$ as Poincare dual classes. This isomorphism of forms and homology classes is used extensively throughout this paper and we often use the same notation for both objects in a Poincare dual pair.

Let X be any oriented manifold of dimension n, A an element of $H_k(X,\mathbb{R})$ and B an element of $H_{n-k}(X,\mathbb{R})$. One can define the intersection number of A and B by taking representative cycles which intersect transversally. The intersection number is then the sum of intersections weighted with a plus or minus sign depending on the orientation of the intersection. In terms of the Poincaré dual forms it is given by

$$A \cdot B = \int_B \eta_A = \int_X \eta_A \wedge \eta_B\,, \tag{A.6}$$

where $\eta_A \in H^{n-k}_{\rm DR}(X,\mathbb{R})$ is the Poincare dual of A and $\eta_B \in H^k_{\rm DR}(X,\mathbb{R})$ is the Poincare dual of B. Since we often denote η_A and η_B by A and B respectively, we can write

$$A \cdot B = \int_X A \wedge B\,. \tag{A.7}$$

Note that $A \cdot B = (-1)^{k(n-k)} B \cdot A$. A non-vanishing intersection number $A \cdot B$ can be positive or negative, depending upon the orientations of the tangent space basis vectors at the points of intersection.

It is frequently essential in this paper to discuss the integer cohomology groups $H_k(X, \mathbb{Z})$. There is a map from $H_k(X, \mathbb{Z}) \to H_k(X, \mathbb{R})$ whose kernel consists of torsion classes. If there is no torsion, the map is an embedding and all of the above statements are correct for $H_k(X, \mathbb{Z})$. If there is torsion, the above formulas still have obvious analogs over $\mathbb{Z}$.

A.2 Effective curves and homology.

Let X be any n-dimensional complex manifold. A curve in X is a closed subset which locally near each of its points can be defined by the vanishing of $n-1$ (and no fewer) holomorphic functions. A curve is irreducible if it is not the union of two proper subsets, each of which is itself a curve. From now on we will take our manifold X to be compact, that is, a complex submanifold of a complex projective space. Then any curve in X is the union of a finite number of irreducible curves. To every curve corresponds its homology class in $H_2(X, \mathbb{Z})$. We say that a class $\mathcal{C}$ is irreducible if it is the class of an irreducible curve (though it may have other representatives which are reducible). We say that a class $\mathcal{C}$ is algebraic if it is a linear combination of irreducible classes with integer coefficients. That is, class $\mathcal{C}$ is algebraic if

$$\mathcal{C} = \sum_i a_i \mathcal{C}_i \,, \tag{A.8}$$

where $\mathcal{C}_i$ are irreducible classes and the coefficients a_i are any integers. Note that when X is a compact manifold the sum is finite. The set of all algebraic classes, denote it by $H_2(X, \mathbb{Z})_{\rm alg}$, forms a subgroup of $H_2(X, \mathbb{Z})$.

A class is called effective if it is algebraic with all the coefficients a_i being non-negative. One can show that there is always a basis of $H_2(X, \mathbb{Z})_{\rm alg}$ composed entirely of effective classes. Clearly, any linear combination of such a basis with non-negative integer coefficients is also an effective class. Note, however, that there can be other effective classes not of this form. In general, the collection of all effective classes forms a cone in $H_2(X, \mathbb{Z})_{\rm alg}$ known as the Mori cone. The Mori cone can be shown to be linearly generated by a set of effective classes. This set includes the effective basis of $H_2(X, \mathbb{Z})_{\rm alg}$ but is, in general, larger. The Mori cone can be finitely generated, as for del Pezzo surfaces, or infinitely generated, as for dP_9 and Enriques surfaces. We refer the reader to appendix B for examples.

By definition, any effective class corresponds to a, in general reducible, curve in X. Non-effective classes can not be interpreted as curves in X, since they involve negative integers. Herein lies the importance of effective classes. For example, in physical applications, such as the five-branes in this paper, it is clearly essential that the classes correspond to curves, as the five-branes must wrap around them. We, therefore, must require five-brane classes to be effective.

B. Complex surfaces

B.1 Properties of del Pezzo surfaces

A del Pezzo surface is a complex manifold of complex dimension two the canonical bundle of which is negative. This means that the dual anticanonical bundle has positive intersection with every curve in the surface. The del Pezzo surfaces which will concern us in this paper are the surfaces dP_r constructed from complex projective space $\mathbb{P}^2$ by blowing up r points $p_1, \ldots, p_r$ in general position where $r = 0, 1, \ldots, 8$.

One also encounters the rational elliptic surface, which we denote dP_9, although it is not a del Pezzo surface in the strict sense. It can be obtained as the blow-up of $\mathbb{P}^2$ at nine points which form the complete intersection of two cubic curves, and which are otherwise in general position. For a dP_9 surface, the anticanonical bundle is no longer positive but, rather, it is "nef", which means that its intersection with every curve on the dP_9 surface is nonnegative. In fact, a dP_9 surface is elliptically fibered over $\mathbb{P}^1$ and the elliptic fibers (the proper transforms of the pencil of cubics through the nine blown up points) are in the anticanonical class. This description fails when the nine points are in completely general position, which is why we require them to be the complete intersection of two (and, hence, of a pencil of) cubics.

Of particular interest is the homology group of curves $H_2(dP_r, \mathbb{Z})$ on the del Pezzo surface. Since a new cycle is created each time a point is blown up, we see that the dimension of $H_2(dP_r, \mathbb{Z})$ is $\dim H_2(dP_r, \mathbb{Z}) = r+1$. From $\mathbb{P}^2$ we thus inherit the single class of hyperplane divisors l. A representative of this class is any linear embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$. The blow-up of the ith point p_i corresponds to an exceptional divisor E_i. Hence, for dP_r, there are r exceptional divisors E_i, $i = 1, \ldots, r$. The curves l and E_i where $i = 1, \ldots, r$ form a basis of homology classes of $H_2(dP_r, \mathbb{Z})$. Note that since dP_r is a rational surface, $H^{2,0}(dP_r) = 0$ and, since on a surface, the Lefschetz theorem relates elements of $H^{1,1}$ to algebraic classes, we have $H_2(dP_r, \mathbb{Z}) = H_2(dP_r, \mathbb{Z})_{\mathrm{alg}}$. A particularly important element of $H_2(dP_r, \mathbb{Z})$ is the anticanonical class $\mathcal{F} = -K_{dP_r}$, given by

$$\mathcal{F} = -K_{dP_r} = 3l - \sum_{i=1}^{r} E_i\,. \tag{B.1}$$

Let us consider the intersection numbers, defined in appendix A, of the basis of curves l and E_i, $i = 1, \ldots, r$ of $H_2(dP_r, \mathbb{Z})$. Now, any two lines in $\mathbb{P}^2$ generically intersect once. Hence one expects, and it can be shown, that

$$l \cdot l = \int_{dP_r} l \wedge l = 1\,. \tag{B.2}$$

It is a known property of the exceptional divisors that each has self intersection number -1. Furthermore, it is clear that exceptional divisors associated with distinct points do not intersect. Therefore, we have

$$E_i \cdot E_j = \int_{dP_r} E_i \wedge E_j = -\delta_{ij} \,. \tag{B.3}$$

Since a general line in $\mathbb{P}^2$ does not pass through any of the blown up points, it follows that the proper transform of a general line in $\mathbb{P}^2$ does not intersect the E_i. Thus, we have

$$E_i \cdot l = \int_{dP_r} E_i \wedge l = 0 \,. \tag{B.4}$$

It is important to explicitly know the set of effective divisors on dP_r. By definition, l and E_i for $i = 1, \ldots, r$ are effective, as is the anticanonical class $\mathcal{F}$. Now consider a line l in $\mathbb{P}^2$ which passes through the ith blown up point p_i. Such a line is still effective. The class of such a curve is given by

$$l - E_i \tag{B.5}$$

and, hence, this is an effective divisor for any $i = 1, \ldots, r$. In general, a line can pass through at most two points, say p_i and p_j where $i \neq j$. The properties of blow-ups then imply that the class of such a curve is

$$l - E_i - E_j \tag{B.6}$$

which, by construction, is an effective divisor for any $i \neq j = 1, \ldots, r$.

In general, a class $\mathcal{C} \in H_2(dP_r, \mathbb{Z})$ is called exceptional if it satisfies

$$\mathcal{C} \cdot \mathcal{C} = -1 \,, \qquad \mathcal{C} \cdot \mathcal{F} = 1 \,, \tag{B.7}$$

where $\mathcal{F}$ is the anticanonical class. The classes of the exceptional curves E_i certainly are of this type, but there are others; for example, the class $l - E_i - E_j$ just described satisfies these properties as well. In fact, any exceptional class on a general del Pezzo surface is the class of a unique, irreducible, non-singular curve which can be blown down without creating any singularities in the resulting surface. This curve is in fact a $\mathbb{P}^1$ and has self-intersection -1. Such curves are called exceptional or simply -1 curves. Even though this is not apparent from our description, all these -1 curves look exactly alike and, in fact, can be interchanged by the Weyl group which acts as a symmetry group of the family of del Pezzo surfaces. So, for example, our del Pezzo surface admits another description, in which the line $l - E_i - E_j$ appears as the blow-up of some point, while one or more of the exceptional divisors E_i appears as a line or higher degree curve.

For $r \leq 4$, all exceptional curves are of the types already discussed. But consider, for $r \geq 5$, a conic in $\mathbb{P}^2$; that is, a curve defined by a quadratic equation. The conic is denoted by $2l$. A conic can pass through at most five blown up points, say p_i, p_j, p_k, p_l, and p_m. If they are all different, then the curves

$$2l - E_i - E_j - E_k - E_l - E_m \tag{B.8}$$

are exceptional divisors. These are easily seen to be effective as well. Similarly, consider a cubic in $\mathbb{P}^2$; that is, a curve defined by a cubic equation. The cubic is denoted by $3l$. When $r = 7$, 8, or 9, a cubic can be chosen to pass through one of the blown up points, say p_i, twice (that is, it will be a singular cubic curve, with singular point at p_i), while also passing (once) through six more of the blown up points, say p_j, p_k, p_l, p_m, p_n, and p_o. Therefore, we see that, for $r = 7$, 8, or 9, we also get exceptional divisors of the form

$$3l - 2E_i - E_j - E_k - E_l - E_m - E_n - E_o\,, \tag{B.9}$$

where all the points are different. Again, these are easily seen to be effective classes. Yet more examples of exceptional curves are obtained, for $r = 8$ or 9, by considering appropriate plane curves of degrees 4, 5, or 6. The complete list of exceptional curves for $r \leq 8$ can be found, for example, in [57, table 3, p. 35]. All these classes are effective.

We can now complete the description of the set of effective classes on a del Pezzo surface. These classes are precisely the linear combinations, with non-negative integer coefficients, of the anticanonical class $\mathcal{F}$ and of the exceptional classes, including the E_i, the curves in (B.6), (B.8), and (B.9), and their more complicated cousins for large r. For $r \leq 8$ this gives us an explicit, finite set which generates the Mori cone.

The above statement, that is, that the effective classes are precisely the linear combinations, with non-negative integer coefficients, of the anticanonical class $\mathcal{F}$ and of the exceptional classes, remains true for the rational elliptic surface dP_9. The new and, perhaps, surprising feature is that on a dP_9 surface there are infinitely many exceptional classes. This is easiest to see using the elliptic fibration structure. Each of the nine exceptional divisors E_i has intersection number 1 with the elliptic fiber $\mathcal{F}$, so it gives a section of the fibration. Conversely, it is easy to see that any section is an exceptional curve. But since each fiber, an elliptic curve, is a group, it follows that the set of sections is itself a group under the operation of pointwise addition of sections. We are free to designate one of our nine sections, say E_9, as the "zero" section. The other eight sections then generate an infinite group of sections, which generically will be $\mathbb{Z}^8$. The Mori cone in this case is not generated by any finite set of effective curves.

Finally, we list the formulas for the Chern classes on dP_r. We find that

$$c_1(dP_r) = -K_{dP_r} = 3l - \sum_{i=1}^{r} E_i \tag{B.10}$$

and

$$c_2(dP_r) = 3 + r \tag{B.11}$$

are the first and second Chern classes of dP_r respectively. The second Chern class is simply a number, since there is only one class in $H_0(dP_r, \mathbb{Z})$.

B.2 Properties of Hirzebruch surfaces

A Hirzebruch surface F is a two-dimensional complex manifold constructed as a fibration with base $\mathbb{P}^1$ and fiber $\mathbb{P}^1$. One way to construct these surfaces is to start with a rank two vector bundle V over $\mathbb{P}^1$ and to take F to be the projectivization of V. For example, we can take V to be

$$V = \mathcal{O} \oplus \mathcal{O}(r)\,, \tag{B.12}$$

where r is a non-negative integer. The resulting Hirzebruch surface is denoted by F_r. It is, in fact, easy to see that all Hirzebruch surfaces arise in this way. We denote the fiber of F_r over $\mathbb{P}^1$ by $\mathcal{E}$. The sections are not all equivalent. Let S_∞ and S_0 denote the two sections of F_r corresponding to the sub-bundles $\mathcal{O}$ and $\mathcal{O}(r)$ respectively.The intersection numbers are found to be

$$\mathcal{E} \cdot \mathcal{E} = 0\,, \qquad S_\infty \cdot S_\infty = -r\,, \qquad S_0 \cdot S_0 = r \tag{B.13}$$

and

$$\mathcal{E} \cdot S_\infty = \mathcal{E} \cdot S_0 = 1\,, \qquad S_\infty \cdot S_0 = 0\,. \tag{B.14}$$

These results are determined as follows. Each section S_∞ and S_0 meets each fiber $\mathcal{E}$ at a unique point, while the self-intersection of a fiber is 0 since it can also be interpreted as the intersection of two distinct, hence disjoint, fibers. The section S_∞ corresponds to the lower degree sub-bundle $\mathcal{O}$, so it can not be moved away from itself. This is reflected in the negative self-intersection number. On the other hand, S_0 corresponds to the larger bundle $\mathcal{O}(r)$. It moves in an r-dimensional linear system and any two representatives meet in r points. But a generic representative of this system does not meet the section at infinity S_∞, thus providing the last intersection number.

We should note, however, that some special curves in the linear system will, in fact, meet S_∞. These are forced to become reducible; that is, they contain S_∞ plus exactly r fibers, leading to the equality

$$S_0 = S_\infty + r\mathcal{E} \tag{B.15}$$

which is valid in $H_2(F_r, \mathbb{Z})$. A basis for $H_2(F_r, \mathbb{Z})$ is provided by $\mathcal{E}$ together with either S_∞ or S_0. The pair $\mathcal{E}$, S_∞ has the advantage that it is also the set of generators for the Mori cone. That is, a class $a\mathcal{E} + bS_\infty$ is effective on F_r for integers a and b if and only if $a \geq 0$ and $b \geq 0$, as is easily seen from the intersection numbers above.

In this paper, we will choose $\mathcal{E}$ and S_∞ as the basis of $H_2(F_r, \mathbb{Z})$. Note again that since F_r is a rational surface, $H^{2,0}(F_r) = 0$ and, hence, $H_2(F_r, \mathbb{Z}) = H_2(F_r, \mathbb{Z})_{\text{alg}}$. If we denote S_∞ simply by S, then the intersection numbers become

$$\mathcal{E} \cdot \mathcal{E} = 0\,, \qquad S \cdot S = -r\,, \qquad S \cdot \mathcal{E} = 1\,. \tag{B.16}$$

The first and second Chern classes are given by

$$c_1(F_r) = 2S + (r+2)\mathcal{E} \tag{B.17}$$

and

$$c_2(F_r) = 4 \tag{B.18}$$

respectively. Finally, to repeat, it is clear that S and $\mathcal{E}$ are effective. Any other irreducible effective curve must have a non-negative intersection number with S and $\mathcal{E}$. From this condition, one finds that all effective curves in F_r are simply linear combinations of S and $\mathcal{E}$ with non-negative coefficients.

B.3 Properties of Enriques surfaces

Following [58], we define an Enriques surface as a complex algebraic surface B with $H^1(B, \mathbb{C}) = 0$, whose canonical bundle is torsion. That is

$$K_B \neq \mathcal{O}_B\,, \qquad K_B \otimes K_B = \mathcal{O}_B\,. \tag{B.19}$$

It follows immediately from the definition that $c_1(B)^2 = 0$ and $h^{2,0} = 0$. The Riemann-Roch theorem then implies that the Euler characteristic is $c_2(B) = 12$, so $h^{1,1} = h^2 = 10$. In fact, the non-trivial cohomology is given by

$$H^2(B, \mathbb{Z}) = \mathbb{Z}^{10} + \mathbb{Z}_2\,. \tag{B.20}$$

That is, the canonical bundle is the only torsion class. The intersection form on $H^2(B, \mathbb{Z})$ vanishes on the torsion, while on the $\mathbb{Z}^{10}$ part it is even, unimodular and of signature (1,9). The torsion canonical bundle implies that the fundamental group of an Enriques surface is nontrivial and, in fact, is $\mathbb{Z}_2$. The universal cover is thus a double cover. It is a surface with Euler characteristic $2 \times 12 = 24$ and it has a trivial canonical bundle. It follows that the universal cover is a K3 surface. In other words, every Enriques surface is obtained as the quotient of a K3 surface by an involution. This involution must act freely, since the K3 is an unramified cover of the Enriques surface. Since we require that the canonical bundle of the Enriques surface is not the trivial bundle, it cannot have any global sections. Thus, the involution on the K3 must send the holomorphic two-form to -1 times itself.

JHEP06(1999)034

Although we will not use this in this paper, we mention the fact that the covering K3 is rather special. Among other properties, it must be elliptically fibered over $\mathbb{P}^1$ and this fibration must also be preserved by the involution. Therefore, the Enriques surface itself inherits a fibration by curves of genus 1. However, we do not consider this to be an elliptic fibration, since it does not have a section. In fact, two of the fibers occur with multiplicity two, which prevents the existence of a section even locally near these fibers. In addition to these two double fibers, there are on a generic Enriques surface exactly 12 singular fibers, just as there are in the elliptic fibration of a dP_9 surface. These two surfaces are actually related, the Enriques surface being obtained from a dP_9 surface by performing logarithmic transforms along the two fibers which thereby become doubled. Conversely, the dP_9 surface can be recovered as the Jacobian fibration of the Enriques surface.

Since $H^{2,0}(B) = 0$, all cohomology two-classes on the Enriques surface are algebraic. We need to decide which of these classes are effective. On a general Enriques surface it turns out that the effective classes fall into two components, each essentially the upper half of a ten-dimensional light cone. First we note that, by the adjunction formula, if $\mathcal{C}$ is an irreducible curve of arithmetic genus g on the Enriques surface, then the self-intersection number is

$$\mathcal{C}^2 = 2g - 2 \geq -2 \tag{B.21}$$

with equality holding if and only if $\mathcal{C}$ is a smooth rational curve. Some special Enriques surfaces may certainly contain such smooth, rational curves, but not the general Enriques surface, as can be seen by a deformation argument. Therefore, we are left to discuss irreducible curves of non-negative self-intersection. Let us ignore torsion for the moment and, hence, consider $H^2(B, \mathbb{R})$. The cone in $H^2(B, \mathbb{R})$ of all classes of non-negative self-intersection looks like the time-like cone of 10D Lorentzian geometry; that is, it consists of two components, the "past" and the "future" (recall that the signature is (1,9)). Any ample class h takes positive values on one side of the cone and negative values on the other. Therefore, all the effective classes are in one half of the cone. Conversely, we claim that all integral classes in this half cone are effective. This follows from the fact that for any class $\mathcal{C}$ with $\mathcal{C}^2 \geq 0$, exactly one of the two classes $\mathcal{C}$ or $-\mathcal{C}$ is effective, as can be seen from the Riemann-Roch formula. Since, in our case, $-\mathcal{C}$ lies in the wrong half cone it cannot be effective and, therefore, $\mathcal{C}$ must be effective.

So far we ignored the torsion by considering $H^2(B, \mathbb{R})$. Returning to $H^2(B, \mathbb{Z}) = \mathbb{Z}^{10} + \mathbb{Z}_2$, we see that along with each class $\mathcal{C}$ comes another class, $K_B + \mathcal{C}$, with the same image in $H^2(B, \mathbb{R})$. Fortunately, with a single exception, these are both effective (or not) together, so the effectivity of a class $\mathcal{C} \in H^2(B, \mathbb{Z})$ depends only on its image in $H^2(B, \mathbb{R})$. The single exception is, of course, the pair 0 and K_B itself. (We are still assuming that our Enriques surface is general, so we only consider classes satisfying $\mathcal{C}^2 \geq 0$.) The reasoning is similar to the above; that is, the Riemann-Roch theorem tells us that either $\mathcal{C}$ or $K_B - \mathcal{C}$ must be effective and, likewise, that either $K_B + \mathcal{C}$ or $-\mathcal{C}$ must be effective. But $\mathcal{C}$ and $-\mathcal{C}$ cannot both be effective (unless $\mathcal{C} = 0$), nor can $K_B - \mathcal{C}$, $K_B + \mathcal{C}$ (unless $\mathcal{C} = K_B$), so if $\mathcal{C}$ is effective, so must be $K_B + \mathcal{C}$ and vice versa.

For more information on Enriques surfaces we refer the reader to [58] or [59].

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JHEP06(1999)034